Gaussian Semiparametric Estimates on the Unit Sphere

Claudio Durastanti
Department of Mathematics, University of Pavia

Xiaohong Lan
Department of Mathematics, University of Connecticut

Domenico Marinucci*
Department of Mathematics, University of Rome "Tor Vergata"

January 16, 2012

Abstract

We study the weak convergence (in the high-frequency limit) of the parameter estimators of power spectrum coefficients associated with Gaussian, spherical and isotropic random fields. In particular, we introduce a Whittle-type approximate maximum likelihood estimator and we investigate its asympotic weak consistency and Gaussianity, in both parametric and semiparametric cases.

- Keywords and phrases: Spherical random fields, high frequency asymptotics, Whittle likelihood, spherical harmonics, parametric and semiparametric estimates
- AMS Classification: 62M15; 62M30; 60G60

1 Introduction

The purpose of this paper is to investigate the asymptotic behavior of a Whittle-like approximate maximum likelihood procedure for the estimation of the spectral parameters (e.g., the *spectral index*) of isotropic Gaussian random fields defined on the unit sphere $\mathbb{S}^2$. In our approach we consider the expansion of the field into spherical harmonics, i.e. we implement a form of Fourier analysis on the sphere, and we implement approximate maximum likelihood estimates under both parametric and semiparametric assumptions on the behavior of the angular power spectrum. We stress that the asymptotic framework we are considering here is rather different from usual - in particular, we assume we are

*Corresponding author, email marinucc@mat.uniroma2.it.

observing a single realization of an isotropic field, the asymptotics being with respect to higher and higher resolution data becoming available (i.e., higher and higher frequency components being observed). In some sense, then the issues we are considering are related to the growing area of fixed-domain asymptotics (see for instance [1], [23]). From the point of view of the proofs, on the other hand, our arguments are in some cases reminiscent of those entertained for instance by [34], where semiparametric estimates of the long memory parameter for covariance stationary processes are analyzed.

In our assumptions, we do not impose a priori a parametric model on the dependence structure of the random field we are analyzing; we rather impose various forms of regularly varying conditions, which only constrain the high-frequency behavior of the angular power spectrum. We are able to show consistency under the least restrictive assumptions; a Central Limit Theorem holds under more restrictive conditions, while asymptotic Gaussianity can be established under general conditions for a slightly-modified (*narrow-band*) procedure, entailing a loss of a logarithmic factor in the rate of convergence. Our analysis is strongly motivated by applications, especially in a Cosmological framework (see e.g. [8], [7]); in this area, huge datasets on isotropic, spherical random fields (usually assumed to be Gaussian) are currently being collected and made publicly available by celebrated satellite missions such as *WMAP* or *Planck* (see for instance http://map.gsfc.nasa.gov/); parameter estimation of the spectral index and other spectral parameters has been considered by many authors (see for instance [13] for a review), but no rigorous asymptotic result has so far been produced, to the best of our knowledge. We thus hope that the consistency and asymptotic Gaussianity properties we provide for our Whittle-like procedure may provide a contribution towards further developments. We refer also to [3], [4], [11], [12], [32], [33], [25] for further theoretical and applied results on angular power spectrum estimation, in a purely nonparametric setting, and to [16], [18], [17], [15], [9], [14], [21], [19], [26] for further results on statistical inference for spherical random fields.

The plan of the paper is as follows: in Section 2, we will recall briefly some well-known background material on harmonic analysis for spherical isotropic random fields; in Section 3 we introduce Whittle-like maximum pseudo-likelihood estimators for angular power spectrum coefficients based on spherical harmonics; Section 4 is devoted to consistency results, while asymptotic Gaussianity is considered in Section 5. In Section 6 we investigate narrow-band estimates, while Section 7 provides some numerical evidence to validate the findings of the paper. Directions for future research are discussed in Section 8, while some auxiliary technical results are collected in the Appendix.

2 Spherical Random Fields and Angular Power Spectrum

In this Section, we will present some well-known background results concerning harmonic analysis on the sphere. We shall focus on zero-mean, isotropic Gaussian random fields $T : \mathbb{S}^2 \times \Omega \to \mathbb{R}$. It is well-known that such fields can be given a spectral representation such that

$$\begin{aligned} T(x) &= \sum_{l \geq 0} \sum_{m=-l}^{l} a_{lm} Y_{lm}(x) = \sum_{l \geq 0} T_l(x) , \qquad (1) \\ a_{lm} &= \int_{\mathbb{S}^2} T(x) \overline{Y}_{lm}(x) \, dx \ . \qquad (2) \end{aligned}$$

where the set of homogenous polynomials $\{Y_{lm} : l \geq 0, m = -l, ..., l\}$ represents an orthonormal basis for the space $L^2(\mathbb{S}^2, dx)$, the class of functions defined on the unitary sphere which are square-integrable with respect to the measure dx (see for instance [35], [14], [26], for more details, and [22], [24] for extensions). Note that this equality holds in both $L^2(\mathbb{S}^2 \times \Omega, dx \otimes \mathbb{P})$ and $L^2(\mathbb{P})$ senses for every fixed $x \in \mathbb{S}^2$. We recall also that for every $g \in SO(3)$ and $x \in \mathbb{S}^2$, a field $T(\cdot)$ is isotropic if and only if

$$T(x) \overset{d}{=} T(gx) \ ,$$

where the equality holds in the sense of processes.

An explicit form for spherical harmonics is given in spherical coordinates $\vartheta \in [0, \pi]$, $\varphi \in [0, 2\pi)$ by:

$$\begin{aligned} Y_{lm}(\vartheta, \varphi) &= \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\vartheta) e^{im\varphi}, \text{ for } m \geq 0 \\ Y_{lm}(\vartheta, \varphi) &= (-1)^m \overline{Y}_{l,-m}(\vartheta, \varphi), \text{ for } m < 0, \end{aligned}$$

$P_{lm}(\cos\vartheta)$ denoting the associated Legendre function; for $m = 0$, we have $P_{l0}(\cos\vartheta) = P_l(\cos\vartheta)$, the standard set of Legendre polynomials (see again [35], [26]). The following orthonormality property holds:

$$\int_{\mathbb{S}^2} Y_{lm}(x) \overline{Y}_{l'm'}(x) \, dx = \delta_l^{l'} \delta_m^{m'} \ .$$

For an isotropic Gaussian field, the spherical harmonics coefficients a_{lm} are Gaussian complex random variables such that

$$\begin{aligned} \mathbb{E}(a_{lm}) &= 0; \\ \mathbb{E}(a_{l_1 m_2} \overline{a}_{l_2 m_2}) &= \delta_{l_2}^{l_1} \delta_{m_2}^{m_1} C_l, \end{aligned}$$

where of course the angular power spectrum C_l fully characterizes the dependence structure under Gaussianity. Further characterizations of the spherical

harmonics coefficients are provided for instance by [2], [26]; here we simply recall that

$$\frac{a_{l0}^2}{C_l} \sim \chi_1^2 \text{ , for } m = 0 \text{ , } \frac{2\left|a_{lm}\right|^2}{C_l} \sim \chi_2^2 \text{ , for } m = \pm 1, \pm 2, ..., \pm l \text{ ,}$$

where all these random variables are independent. Given a realization of the random field, an estimator of the angular power spectrum can be defined as:

$$\widehat{C}_l = \frac{1}{2l+1} \sum_{m=-l}^{l} \left|a_{lm}\right|^2 , \tag{3}$$

the so-called empirical angular power spectrum. It is immediately seen that

$$\mathbb{E}\widehat{C}_l = \frac{1}{2l+1} \sum_{m=-l}^{l} C_l = C_l \text{ , } Var\left(\frac{\widehat{C}_l}{C_l}\right) = \frac{2}{2l+1} \rightarrow 0 \text{ for } l \rightarrow +\infty.$$

It can be convenient to write $\widetilde{C}_l = \widehat{C}_l / C_l$.

We shall now focus on some semiparametric models on the angular power spectrum; here, by semiparametric we mean that we shall assume a parametric form on the asymptotic behavior of C_l, but we shall refrain from a full characterization over all multipoles l. More precisely, we formulate the following:

Condition 1 *The random field $T(x)$ is Gaussian and isotropic with angular power spectrum such that:*

$$C_l(\vartheta) = C_l(G, \alpha_0) = G(l)\, l^{-\alpha_0} > 0, \tag{4}$$

where $\alpha_0 \in (2, +\infty) \equiv A$ and for all $l = 1, 2, ...$

$$0 < c_1 \leq G(l) \leq c_2 < +\infty \text{ .}$$

Condition A seems very mild, as it is basically requiring only some form of regular variation on the tail behavior of the angular power spectrum C_l. For instance, in the CMB framework the so-called *Sachs-Wolfe* power spectrum (i.e. the leading model for fluctuations of the primordial gravitational potential) takes the form (4), the spectral index α_0 capturing the scale invariance properties of the field itself (α_0 is expected to be close to 2 from theoretical considerations, a prediction so far in good agreement with observations, see for instance [8] and [20]). For our asymptotic results below, we shall need to strengthen it somewhat; as we shall see, Condition 2 will turn out to be sufficient to establish a rate of convergence for our estimator, under Condition 3 we will be able to provide a Law of Large Numbers, while under Condition 4 our estimates will be shown to be asymptotically Gaussian and centered, thus making statistical inference feasible. On the other hand, in Section 6 we shall be able to provide narrow-band estimates with asymptotically centred limiting Gaussian law under Condition 2, to the price of a logarithmic term in the rate of convergence. Of course, the conditions below are nested, i.e. 4 implies 3, which trivially implies 2.

Condition 2 *Condition 1 holds and moreover, $G(l)$ satisfies the smoothness condition*

$$G(l) = G_0 \left\{ 1 + O(\frac{1}{l}) \right\} \ ,$$

Condition 3 *Condition 2 holds and moreover, $G(l)$ satisfies*

$$G(l) = G_0 \left\{ 1 + \frac{\kappa}{l} + o(\frac{1}{l}) \right\} \ ,$$

Condition 4 *Condition 3 holds with $\kappa = 0$, i.e. $G(l)$ satisfies the smoothness condition*

$$G(l) = G_0 \left\{ 1 + o(\frac{1}{l}) \right\} \ .$$

Example 5 *As straightforward examples that satisfy the previous assumptions, consider the rational function*

$$G\left(l\right) = \frac{\Pi_1\left(l\right)}{\Pi_2\left(l\right)} \ , \tag{5}$$

where $\Pi_1(l)$ and $\Pi_2\left(l\right)$ are positive valued polynomials of order $k \in \mathbb{N}$ and satisfying:

$$0 < c_1 \leq \frac{\Pi_1\left(l\right)}{\Pi_2\left(l\right)} \leq c_2 < +\infty \ .$$

We have that

$$\begin{aligned}
\frac{\Pi_1\left(l\right)}{\Pi_2\left(l\right)} &= \frac{p_0 + p_1 l + \ldots p_k l^k}{q_0 + q_1 l + \ldots q_k l^k} \\
&= \frac{p_k}{q_k} \left\{ \frac{\frac{p_0}{p_k}\frac{1}{l^k} + \ldots + \frac{p_{k-2}}{p_k}\frac{1}{l^2} + \frac{p_{k-1}}{p_k}\frac{1}{l} + 1}{\frac{q_0}{q_k}\frac{1}{l^k} + \ldots + \frac{q_{k-2}}{q_k}\frac{1}{l^2} + \frac{q_{k-1}}{q_k}\frac{1}{l} + 1} \right\} \\
&= \frac{p_k}{q_k} \left\{ 1 + \frac{\left(\frac{p_0}{p_k} - \frac{q_0}{q_k}\right)\frac{1}{l^k} + \ldots + \left(\frac{p_{k-2}}{p_k} - \frac{q_{k-2}}{q_k}\right)\frac{1}{l^2} + \left(\frac{p_{k-1}}{p_k} - \frac{q_{k-1}}{q_k}\right)\frac{1}{l}}{\frac{q_0}{q_k}\frac{1}{l^k} + \ldots + \frac{q_{k-2}}{q_k}\frac{1}{l^2} + \frac{q_{k-1}}{q_k}\frac{1}{l} + 1} \right\} \ ,
\end{aligned}$$

whence

$$C_l = \frac{p_k}{q_k} l^{-\alpha_0} \left\{ 1 + \left(\frac{p_{k-1}}{p_k} - \frac{q_{k-1}}{q_k} \right) \frac{1}{l} + O(\frac{1}{l^2}) \right\} \ . \tag{6}$$

It follows that (5) satisfies Condition 3 (and hence 2) for

$$G_0 = \frac{p_k}{q_k} \text{ and } \kappa = \frac{p_{k-1}}{p_k} - \frac{q_{k-1}}{q_k} \ ;$$

Condition 4 is satisfied when $p_{k-1} = q_{k-1} = 0$, as for instance in

$$C_l = l^{-\alpha_0} \left\{ \frac{1 + l^2}{1 + 2l^2} \right\} = \frac{l^{-\alpha_0}}{2} \left\{ 1 + \frac{1}{1 + 2l^2} \right\}$$

or, more generally, for $\frac{p_{k-1}}{p_k} = \frac{q_{k-1}}{q_k}$, as for instance

$$C_l = l^{-\alpha_0} \left\{ \frac{1 + l + 2l^2}{1 + 2l + 4l^2} \right\} = \frac{l^{-\alpha_0}}{2} \left\{ 1 + \frac{1}{1 + 2l + 4l^2} \right\} \ .$$

3 A Whittle-like approximation to the likelihood function

Our aim in this Section is to discuss heuristically a Whittle-like approximation for the log-likelihood of isotropic spherical Gaussian fields, and to derive the corresponding estimator. Assume that the triangular array $\{a_{lm}\}$, $m = -l, ..., l$, $l = 1, 2, ...L$ is evaluated from the observed field $\{T(x)\}$, by means of (2). Our motivating rationale is the idea that a set of harmonic components up to multipole L can be reconstructed without observational noise or numerical error, whereas the following are simply discarded; this is clearly a simplified picture, but we believe it provides an accurate approximation to many current experimental set-ups. Of course, L grows larger when more sophisticated experiments are run (L can be considered in the order of 500/600 for data collected from $WMAP$ and 1500/2000 for those from $Planck$). It is readily seen from (3) that

$$\widehat{C}_l = a_{l0}^2 + 2\sum_{m=1}^{l} \left[\operatorname{Re}\{a_{lm}\}\right]^2 + 2\sum_{m=1}^{l} \left[\operatorname{Im}\{a_{lm}\}\right]^2 \ ,$$

where the variables $\left\{a_{l0}, \sqrt{2}\operatorname{Re}\{a_{l1}\}, \sqrt{2}\operatorname{Im}\{a_{l1}\}, ..., \sqrt{2}\operatorname{Re}\{a_{ll}\}, \sqrt{2}\operatorname{Im}\{a_{ll}\}\right\}$ are $i.i.d.$ Gaussian variables with law $\mathcal{N}(0, C_l)$, see [2]. The likelihood function can then be written down as

$$-2\log \mathcal{L}_l(\vartheta; \{a_{lm}\}_{m=-l}^{l}) = (2l+1)\log(2\pi) + (2l+1)\log C_l(\vartheta) + (2l+1)\frac{\widehat{C}_l}{C_l(\vartheta)} \ ,$$

or equivalently

$$= const + (2l+1)\frac{\widehat{C}_l}{C_l(\vartheta)} - (2l+1)\log\frac{\widehat{C}_l}{C_l(\vartheta)} \ .$$

Clearly this landscape is overly simplified, for instance, due to numerical errors and aliasing effects the expected value $\mathbb{E}\left|a_{lm}\right|^2$ may not be exactly equal to the population model $C_l(\vartheta)$; however in Conditions 1 and following we are allowing the two to differ to various degrees, and we expect this to cover to some of effect these experimental features that we are neglecting. More generally, rather than a sharp cutoff at L, a smooth transition towards noisier frequencies would represent more efficiently actual experimental circumstances, but we view our results here as a first step amenable of further developments.

An alternative heuristics for our framework can be introduced considering that for $l = 1, 2, ..., L$, the following Fourier components can be observed on a discrete grid of points $\{x_1, ..., x_K\}$

$$\overrightarrow{T}_l = \{T_l(x_1), ..., T_l(x_k), ..., T_l(x_K)\} \ .$$

To simplify our discussion, we shall also pretend that $\{x_1, ..., x_K\}$ form a set of approximate *cubature points* with constant *cubature weights* $\lambda_k = 4\pi/K$ (see

for instance [28], [29]), so that we have

$$\sum_k \frac{4\pi}{K} Y_{lm_1}(x_k)\overline{Y}_{lm_2}(x_k) \simeq \delta_{m_1}^{m_2} \text{ , for } l = 1, 2, ..., L \ .$$

Again this landscape is overly simplified, for instance, cubature weights on the sphere are known not to be constant, but their variation is usually considered numerically negligible. The frequency components T_l are well-known to be independent and we can hence write down the likelihood function as

$$\mathcal{L}(\vartheta; T) := \prod_{l=1}^{L} \mathcal{L}_l(\vartheta; \overrightarrow{T}_l) \ ,$$

where

$$\mathcal{L}_l(\vartheta; \overrightarrow{T}_l) = (2\pi)^{-(2l+1)/2} \Omega_l^{-1/2} \exp\left\{-\frac{1}{2}\overrightarrow{T}_l' \Omega_l^{-1} \overrightarrow{T}_l\right\} \ ,$$

$$\{\Omega_l\}_{jk} = \left\{\Omega_l(x_j, x_k) = \frac{2l+1}{4\pi} C_l P_l(\langle x_j, x_k \rangle)\right\} \ .$$

The matrix Ω_l can be (approximately) decomposed as follows:

$$\Omega_l \simeq \sqrt{\frac{4\pi}{K}} \begin{bmatrix} Y_{l,-l}(x_1) & Y_{l,-l+1}(x_1) & ... & Y_{l,l-1}(x_1) & Y_{l,l}(x_1) \\ Y_{l,-l}(x_2) & ... & ... & ... & Y_{l,l}(x_2) \\ .... & ... & ... & ... & ... \\ Y_{l,-l}(x_{K-1}) & ... & ... & ... & Y_{l,l}(x_{K-1}) \\ Y_{l,-l}(x_K) & Y_{l,-l+1}(x_K) & ... & Y_{l,l-1}(x_K) & Y_{l,l}(x_K) \end{bmatrix}$$

$$\times \frac{K}{4\pi} C_l I_{2l+1} \times \sqrt{\frac{4\pi}{K}} \begin{bmatrix} \overline{Y}_{l,-l}(x_1) & \overline{Y}_{l,-l}(x_2) & ... & \overline{Y}_{l,-l}(x_{K-1}) & \overline{Y}_{l,-l}(x_K) \\ \overline{Y}_{l,-l+1}(x_1) & ... & ... & ... & \overline{Y}_{l,-l+1}(x_K) \\ .... & ... & ... & ... & ... \\ \overline{Y}_{l,l-1}(x_1) & ... & ... & ... & \overline{Y}_{l,l-1}(x_K) \\ \overline{Y}_{l,l}(x_1) & \overline{Y}_{l,l}(x_2) & ... & \overline{Y}_{l,l}(x_{K-1}) & \overline{Y}_{l,l}(x_K) \end{bmatrix}$$

$$=: \mathcal{Y}_l \times C_l(\vartheta) I_{2l+1} \times \mathcal{Y}_l^* \ .$$

In fact

$$\mathcal{Y}_l^* \mathcal{Y}_l \simeq I_{2l+1} \text{ and } \det\{\Omega_l\} \simeq C_l^{2l+1}(\vartheta) \ .$$

Hence

$$-2 \log \mathcal{L}_l(\vartheta; \overrightarrow{T}_l) \simeq K + (2l+1) \log C_l(\vartheta) + \left\{\overrightarrow{T}_l' \mathcal{Y}_l \times C_l^{-1}(\vartheta) I_{2l+1} \times \mathcal{Y}_l^* \overrightarrow{T}_l\right\} \ .$$

Now

$$
\mathcal{Y}_l^* \overrightarrow{T}_l = \sqrt{\frac{4\pi}{K}} \begin{bmatrix} \overline{Y}_{l,-l}(x_1) & \overline{Y}_{l,-l}(x_2) & ... & \overline{Y}_{l,-l}(x_{K-1}) & \overline{Y}_{l,-l}(x_K) \\ \overline{Y}_{l,-l+1}(x_1) & ... & ... & ... & \overline{Y}_{l,-l+1}(x_K) \\ .... & ... & ... & ... & ... \\ \overline{Y}_{l,l-1}(x_1) & ... & ... & ... & \overline{Y}_{l,l-1}(x_K) \\ \overline{Y}_{l,l}(x_1) & \overline{Y}_{l,l}(x_2) & ... & \overline{Y}_{l,l}(x_{K-1}) & \overline{Y}_{l,l}(x_K) \end{bmatrix}
$$

$$
\times \begin{bmatrix} \sum_m a_{lm} Y_{lm}(x_1) \\ \sum_m a_{lm} Y_{lm}(x_2) \\ .... \\ \sum_m a_{lm} Y_{lm}(x_{K-1}) \\ \sum_m a_{lm} Y_{lm}(x_K) \end{bmatrix}
$$

$$
= \sqrt{\frac{4\pi}{K}} \begin{bmatrix} \sum_{m_1} a_{lm_1} \sum_k Y_{lm_1}(x_k) \overline{Y}_{l,-l}(x_k) \\ \sum_{m_1} a_{lm_1} \sum_k Y_{lm_1}(x_k) \overline{Y}_{l,-l+1}(x_k) \\ .... \\ \sum_{m_1} a_{lm_1} \sum_k Y_{lm_1}(x_k) \overline{Y}_{l,l-1}(x_k) \\ \sum_{m_1} a_{lm_1} \sum_k Y_{lm_1}(x_k) \overline{Y}_{l,l}(x_k) \end{bmatrix} \simeq \sqrt{\frac{K}{4\pi}} \begin{bmatrix} a_{l,-l} \\ a_{l,-l+1} \\ .... \\ a_{l,l-1} \\ a_{l,l} \end{bmatrix} ,
$$

whence

$$
\left\{ \overrightarrow{T}'_l \mathcal{Y}_l \times \frac{4\pi}{K} \frac{1}{C_l(\vartheta)} I_{2l+1} \times \mathcal{Y}_l^* \overrightarrow{T}_l \right\} \simeq \sum_m \frac{|a_{lm}|^2}{C_l(\vartheta)} = (2l+1) \frac{\widehat{C}_l}{C_l(\vartheta)} .
$$

As before, we can then conclude heuristically that

$$
-2 \log \mathcal{L}_l(\vartheta; \overrightarrow{T}_l) \simeq const + (2l+1) \frac{\widehat{C}_l}{C_l(\vartheta)} - (2l+1) \log \frac{\widehat{C}_l}{C_l(\vartheta)} . \tag{7}
$$

Again we stress that for a general spherical random field with an infinite-terms expansion such as (1) the relationship (7) cannot hold exactly; indeed, precise cubature formulae can be established only for finite order spherical harmonics. In general, this may introduce some numerical error: as mentioned before, however, we pretend in this paper that such correction factors are covered by Conditions 1-4. In other words, we envisage a situation where data analysis is carried over on multipoles l where numerical errors are of smaller order and the approximation (4) holds for the expected variance of the sample $\{a_{lm}\}$.

4 Asymptotic results: Consistency

As motivated in the Introduction, in this paper we shall not assume we have actually available a fully parametric model for the angular power spectrum. Instead, the idea will be to use an approximate maximum likelihood estimator, which shall exploit the asymptotic approximation provided by Condition 1, i.e. $C_l \simeq G l^{-\alpha}$. In view of the discussion in the previous Section, the following Definition seems rather natural:

Definition 6 *The* Spherical Whittle estimator *for the parameters* (α_0, G_0) *is provided by*

$$(\widehat{\alpha}_L, \widehat{G}_L) := \arg \min_{\alpha \in A, G \in \Gamma} \sum_{l=1}^{L} \left\{ (2l+1) \frac{\widehat{C}_l}{Gl^{-\alpha}} - (2l+1) \log \frac{\widehat{C}_l}{Gl^{-\alpha}} \right\} . \quad (8)$$

Remark 7 *For general parametric models* $C_l = C_l(\vartheta)$, *the* Spherical Whittle estimator *for a parameter* $\vartheta \in \Theta \subset \mathbb{R}^p$ *can be obviously defined as*

$$\widehat{\vartheta}_L := \arg \min_{\vartheta \in \Theta} \sum_{l=1}^{L} \left\{ (2l+1) \frac{\widehat{C}_l}{C_l(\vartheta)} - (2l+1) \log \frac{\widehat{C}_l}{C_l(\vartheta)} \right\} . \quad (9)$$

Remark 8 *To ensure that the estimator exists, as usual we shall assume throughout this paper that the parameter space is a compact subset of* $A \times \Gamma \in \mathbb{R}^2$; *more precisely we take* $\alpha \in A = [a_1, a_2]$, $2 < a_1 < a_2 < \infty$, *and* $G \in \Gamma = [\gamma_1, \gamma_2]$, $0 < \gamma_1 < \gamma_2 < \infty$. *This is little more than a formal requirement that is standard in the literature on (pseudo-)maximum likelihood estimation.*

We can rewrite in a more transparent form the previous estimator following an argument analogous to [34], i.e. "concentrating out" the parameter G. Indeed, the previous minimization problem is equivalent to let us consider

$$\begin{aligned} (\widehat{\alpha}_L, \widehat{G}_L) \quad &: \quad = \arg \min_{\alpha, G} \mathcal{R}_L(G, \alpha) \\ \mathcal{R}_L(G, \alpha) \quad &: \quad = \sum_{l=1}^{L} (2l+1) \frac{\widehat{C}_l}{Gl^{-\alpha}} + \sum_{l=1}^{L} (2l+1) \log G + \sum_{l=1}^{L} (2l+1) \log l^{-\alpha}. \end{aligned}$$

It is readily seen that:

$$\frac{\partial \mathcal{R}_L(G, \alpha)}{\partial G} = -\frac{1}{G^2} \sum_{l=1}^{L} (2l+1) \frac{\widehat{C}_l}{l^{-\alpha}} + \frac{1}{G} \sum_{l=1}^{L} (2l+1)$$

so that

$$\frac{\partial \mathcal{R}_L(G, \alpha)}{\partial G} = 0 \Longleftrightarrow G = \widehat{G}(\alpha) := \frac{1}{\sum_{l=1}^{L} (2l+1)} \sum_{l=1}^{L} (2l+1) \frac{\widehat{C}_l}{l^{-\alpha}} .$$

Moreover, we have

$$\begin{aligned}
\frac{\partial^2 \mathcal{R}_L(G,\alpha)}{\partial G^2} &= 2\frac{1}{G^3}\sum_{l=1}^{L}(2l+1)\frac{\widehat{C}_l}{l^{-\alpha}} - \frac{\sum_{l=1}^{L}(2l+1)}{G^2} \\
&= \frac{1}{G^2}\left[2\frac{1}{G}\sum_{l=1}^{L}(2l+1)\frac{\widehat{C}_l}{l^{-\alpha}} - \sum_{l=1}^{L}(2l+1)\right] \\
\left.\frac{\partial^2 \mathcal{R}_L(G,\alpha)}{\partial G^2}\right|_{G=\widehat{G}(\alpha)} &= \frac{1}{\left|\widehat{G}(\alpha)\right|^2} 2\frac{\sum_{l=1}^{L}(2l+1)}{\sum_{l=1}^{L}(2l+1)\frac{\widehat{C}_l}{l^{-\alpha}}}\sum_{l=1}^{L}(2l+1)\frac{\widehat{C}_l}{l^{-\alpha}} - \sum_{l=1}^{L}(2l+1) \\
&= \frac{1}{\left|\widehat{G}(\alpha)\right|^2}\sum_{l=1}^{L}(2l+1) > 0.
\end{aligned}$$

Considering, moreover, that on the boundary of $A \times \Gamma$, $\mathcal{R}_L(G,\alpha) = +\infty$, $G \to \mathcal{R}_L(G,\alpha)$ has a unique minimum over its domain on $G = \widehat{G}(\alpha)$. Thus we can consider:

$$\begin{aligned}
\widehat{\alpha}_L &= \arg\min_{\alpha} R_L(\alpha), \qquad (10) \\
R_L(\alpha) &= \frac{\mathcal{R}_L(\widehat{G}(\alpha),\alpha)}{\sum_{l=1}^{L}(2l+1)} - 1 = \\
&= \left(\log \widehat{G}(\alpha) - \frac{\alpha}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\log l\right).
\end{aligned}$$

Our first result is the following

Theorem 9 *Under Condition 1, as $L \to \infty$ we have (compare (6))*

$$\widehat{\alpha}_L \to_p \alpha_0 \ ;$$

moreover, under Condition 2,

$$\widehat{G}_L \to_p G_0 \ .$$

Proof. To establish consistency, we shall resort to a technique developed by [6] and [34]. In particular, let us now write

$$\Delta R_L(\alpha,\alpha_0) = R_L(\alpha) - R_L(\alpha_0) =$$

$$= \log\frac{\widehat{G}(\alpha)}{G(\alpha)} - \log\frac{\widehat{G}(\alpha_0)}{G(\alpha_0)} - \frac{(\alpha-\alpha_0)}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\log l + \log\frac{G(\alpha)}{G(\alpha_0)} \ ,$$

where

$$\begin{aligned}
G(\alpha) &= \frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\frac{G_0 l^{-\alpha_0}}{l^{-\alpha}} \ , \ G(\alpha_0) = G_0 \ , \\
\log\frac{G(\alpha)}{G(\alpha_0)} &= \log\left\{\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)l^{\alpha-\alpha_0}\right\} .
\end{aligned}$$

so that

$$\Delta R_L(\alpha, \alpha_0) = U_L(\alpha, \alpha_0) - T_L(\alpha, \alpha_0),$$

$$U_L(\alpha, \alpha_0) = -\frac{(\alpha - \alpha_0)}{\sum_{l=1}^{L}(2l+1)} \sum_{l=1}^{L} (2l+1) \log l + \log \frac{G(\alpha)}{G(\alpha_0)} \tag{11}$$

$$T_L(\alpha, \alpha_0) = \log \frac{\widehat{G}(\alpha_0)}{G(\alpha_0)} - \log \frac{\widehat{G}(\alpha)}{G(\alpha)} \; . \tag{12}$$

The proof is then completed with the aid of the auxiliary Lemmas 11, 12 that we shall discuss below. Indeed

$$\begin{aligned} \Pr\{|\widehat{\alpha}_L - \alpha_0| > \varepsilon\} &\leq \Pr\left\{\inf_{|\alpha-\alpha_0|>\varepsilon} \Delta R_L(\alpha, \alpha_0) \leq 0\right\} \\ &\leq \Pr\left\{\inf_{|\alpha-\alpha_0|>\varepsilon} [U_L(\alpha, \alpha_0) - T_L(\alpha, \alpha_0)] \leq 0\right\} \; . \end{aligned}$$

For $\alpha_0 - \alpha < 2$ the previous probability is bounded by, for any $\delta > 0$

$$\leq \Pr\left\{\inf_{|\alpha-\alpha_0|>\varepsilon} U_L(\alpha, \alpha_0) \leq \delta\right\} + \Pr\left\{\sup_{|\alpha-\alpha_0|>\varepsilon} T_L(a, \alpha_0) > 0\right\} \; ,$$

and

$$\lim_{L\to\infty} \Pr\left\{\sup_{|\alpha-\alpha_0|>\varepsilon} T_L(a, \alpha_0) > 0\right\} = 0$$

from Lemma 12, while from Lemma 11 there exist $\delta_\varepsilon = (1 + \varepsilon/2) - \log(1 + \varepsilon/2) - 1 > 0$ such that

$$\lim_{L\to\infty} \Pr\left\{\inf_{|\alpha-\alpha_0|>\varepsilon} U_L(\alpha, \alpha_0) \leq \delta_\varepsilon\right\} = 0 \; .$$

For $\alpha_0 - \alpha = 2$ or $\alpha_0 - \alpha > 2$ the same result is obtained by dividing $\Delta R_L(\alpha, \alpha_0)$ by, respectively $\log \log L$ or $\log L$ and then resorting again to Lemmas 11, 12.

Now note that

$$\begin{aligned} \widehat{G}(\widehat{\alpha}_L) - G_0 &= \frac{1}{\sum_{l=1}^{L}(2l+1)} \sum_{l=1}^{L} (2l+1) \frac{\widehat{C}_l}{l^{-\widehat{\alpha}_L}} - \frac{1}{\sum_{l=1}^{L}(2l+1)} \sum_{l=1}^{L} (2l+1) \frac{G_0 l^{-\alpha_0}}{l^{-\alpha_0}} = \\ &= \frac{G_0}{\sum_{l=1}^{L}(2l+1)} \sum_{l=1}^{L} (2l+1) l^{-\alpha_0+\widehat{\alpha}_L} \left(\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - \frac{l^{-\widehat{\alpha}_L}}{l^{-\alpha_0}}\right) = \\ &= \frac{1}{\sum_{l=1}^{L}(2l+1)} \sum_{l=1}^{L} (2l+1) G_0 l^{-(\alpha_0-\widehat{\alpha}_L)} \left\{\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - l^{(\alpha_0-\widehat{\alpha}_L)}\right\} = \\ &= \frac{1}{\sum_{l=1}^{L}(2l+1)} \sum_{l=1}^{L} (2l+1) G_0 l^{-(\alpha_0-\widehat{\alpha}_L)} \left\{\left(\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - 1\right) + \left(1 - l^{(\alpha_0-\widehat{\alpha}_L)}\right)\right\}. \end{aligned}$$

Clearly:

$$\begin{aligned}\left|\widehat{G}(\widehat{\alpha}_L)-G_0\right| &\leq \left|\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)G_0 l^{-(\alpha_0-\widehat{\alpha}_L)}\left\{\left(\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}}-1\right)\right\}\right|+\\ &+\left|\frac{G_0}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\left(l^{-(\alpha_0-\widehat{\alpha}_L)}-1\right)\right|\\ &= |G_A|+|G_B|\,,\end{aligned}$$

so that

$$\Pr\left(\left|\widehat{G}(\widehat{\alpha}_L)-G_0\right|\geq\varepsilon\right)\leq\Pr\left(|G_A|\geq\frac{\varepsilon}{2}\right)+\Pr\left(|G_B|\geq\frac{\varepsilon}{2}\right)\ .$$

Observe that:

$$\Pr\left\{|G_A|\geq\frac{\varepsilon}{2}\right\}\leq\Pr\left\{\left[|G_A|\geq\frac{\varepsilon}{2}\right]\cap\left[|\alpha_0-\widehat{\alpha}_L|<\frac{1}{3}\right]\right\}$$
$$+\Pr\left\{|\alpha_0-\widehat{\alpha}_L|\geq\frac{1}{3}\right\}$$

$$\begin{aligned}&\leq \Pr\left\{\left[\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)G_0 l^{1/3}\left|\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}}-1\right|\geq\varepsilon\right]\right\}+o_L(1)\\ &\leq \frac{1}{\varepsilon}\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)G_0 l^{1/3}\mathbb{E}\left|\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}}-1\right|+o_L(1)\\ &\leq \frac{C}{\varepsilon}\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)G_0 l^{1/3}l^{-1/2}+o_L(1)\\ &= \frac{C}{\varepsilon}\frac{L^{11/6}}{\sum_{l=1}^{L}(2l+1)}+o_L(1)=o_L(1)\ .\end{aligned}$$

As far as the second term is concerned, we have, for a suitably small $\delta>0$:

$$\begin{aligned}\Pr\left(|G_B|\geq\frac{\varepsilon}{2}\right) &= \Pr\left(\left[|G_B|\geq\frac{\varepsilon}{2}\right]\cap\left[\log l\,(\alpha_0-\widehat{\alpha}_L)\right]<\delta\right)+\Pr\left(\log l\,(\alpha_0-\widehat{\alpha}_L)\geq\delta\right)\\ &= \Pr\left(\left[|G_B|\geq\frac{\varepsilon}{2}\right]\cap\left[\log l\,(\alpha_0-\widehat{\alpha}_L)\right]<\delta\right)+o_L(1)\ .\end{aligned}$$

and using $|e^{-x}-1|\leq x$ for $0\leq x\leq 1$, we obtain

$$\left|l^{-(\alpha_0-\widehat{\alpha}_L)}-1\right|=|\exp(-\log l\,(\alpha_0-\widehat{\alpha}_L)-1|\leq\log l\,|\alpha_0-\widehat{\alpha}_L|\ ,$$

$$\Pr\left(\left[|G_B|\geq\frac{\varepsilon}{2}\right]\cap\left[\log l\,(\alpha_0-\widehat{\alpha}_L)\right]<\delta\right)$$

$$
\begin{aligned}
&\leq \Pr\left(\frac{G_0}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\left|\left(l^{-(\alpha_0-\widehat{\alpha}_L)}-1\right)\right| \geq \frac{\varepsilon}{2} \cap \left[\log l\left(\alpha_0-\widehat{\alpha}_L\right)\right] < \delta\right) \\
&\leq \frac{1}{\varepsilon}\mathbb{E}\left\{\frac{G_0}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\log l\left|\alpha_0-\widehat{\alpha}_L\right|\right\} \\
&\leq \frac{C}{\varepsilon}\frac{G_0}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\log l\frac{\log L}{L} = o_L(1)\ ,
\end{aligned}
$$

where we have used

$$
\mathbb{E}\left|\alpha_0-\widehat{\alpha}_L\right| \leq \left\{\mathbb{E}\left|\alpha_0-\widehat{\alpha}_L\right|^2\right\}^{1/2} = O(\frac{\log L}{L})\ ,
$$

which under Condition 2 will be established in the proof of Theorem 16. ■

The first auxiliary result we shall need concerns $G, \widehat{G}$ and their k-th order derivatives $G_k, \widehat{G}_k$, i.e.

$$
\widehat{G}_k\left(\alpha\right) = \frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k\frac{\widehat{C}_l}{l^{-\alpha}}\ ,\ k=0,1,2,...
$$

$$
G_k\left(\alpha\right) = \frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k\frac{G_0 l^{-\alpha_0}}{l^{-\alpha}}\ ,\ k=0,1,2,...
$$

where $\widehat{G}_0(\alpha) = \widehat{G}(\alpha)$ and $G_0(\alpha) = G(\alpha)$ defined as above.

Lemma 10 *Under Condition 2, for all* $2 > \alpha_0 - \alpha > \varepsilon > 0$, *as* $L \to \infty$, *we have*

$$
\sup_{\alpha}\left|\log\frac{\widehat{G}_k(\alpha)}{G_k(\alpha)}\right| = o_p(1)\ .
$$

On the other hand, if $\alpha_0 - \alpha \geq 2$,

$$
\sup_{\alpha}\left|\log\frac{\widehat{G}_k(\alpha)}{G_k(\alpha)}\right| = O_p(1)\ .
$$

Proof. Let us first focus on the case where $\alpha - \alpha_0 > -2$. For clarity of exposition, we start from a simplified parametric version of Condition 1, i.e. we assume that we have exactly

$$
C_l(\vartheta) = C_l(G_0, \alpha_0) = G_0 l^{-\alpha_0}. \tag{13}
$$

Let us write first

$$\begin{aligned}
\frac{\widehat{G}_k(\alpha)}{G_k(\alpha)} - 1 &= \frac{\frac{1}{\sum_{l=1}^{L}(2l+1)} \sum_{l=1}^{L}(2l+1)\left(\log l\right)^k \frac{\widehat{C}_l}{l^{-\alpha}}}{\frac{1}{\sum_{l=1}^{L}(2l+1)} \sum_{l=1}^{L}(2l+1)\left(\log l\right)^k \frac{G_0 l^{-\alpha_0}}{l^{-\alpha}}} - 1 \\
&= \frac{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k \frac{\widehat{C}_l}{l^{-\alpha}} - \sum_{l=1}^{L}(2l+1)\left(\log l\right)^k \frac{G_0 l^{-\alpha_0}}{l^{-\alpha}}}{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k \frac{G_0 l^{-\alpha_0}}{l^{-\alpha}}} \\
&= \frac{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k G_0 l^{\alpha-\alpha_0}\left\{\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - 1\right\}}{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k G_0 l^{\alpha-\alpha_0}} \ .
\end{aligned}$$

Fixed $0 < \beta < \frac{1}{2}$, we have, for all l :

$$\Pr\left(\left|\frac{\sum_{l=1}^{L}(2l+1) G_0 l^{\alpha-\alpha_0}\left(\log l\right)^k \left\{\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - 1\right\}}{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k G_0 l^{\alpha-\alpha_0}}\right| > \delta_\varepsilon\right)$$

$$\begin{aligned}
&\leq \Pr\left(L^\beta \left|\frac{\sum_{l=1}^{L}\sqrt{(2l+1)}\left(\log l\right)^k l^{\alpha-\alpha_0}}{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k l^{\alpha-\alpha_0}}\right| \frac{\sup_l \sqrt{(2l+1)}\left|\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - 1\right|}{L^\beta} > \delta_\varepsilon\right) \\
&\leq \Pr\left(\sup_l \sqrt{(2l+1)}\left|\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - 1\right| > \delta_\varepsilon L^\beta\right) \ .
\end{aligned}$$

because

$$L^\beta \frac{\sum_{l=1}^{L}\sqrt{(2l+1)}\left(\log l\right)^k l^{\alpha-\alpha_0}}{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k l^{\alpha-\alpha_0}} = C\frac{L^{\beta+\frac{3}{2}+\alpha-\alpha_0}\log^k L}{L^{2+\alpha-\alpha_0}\log^k L} = CL^{\beta-1/2} = o(1) \ .$$

Now

$$\begin{aligned}
&\Pr\left\{\sup_l \sqrt{(2l+1)}\left|\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - 1\right| > \delta_\varepsilon L^\beta\right\} \\
\leq\ & L \max_l \Pr\left\{\sqrt{(2l+1)}\left|\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - 1\right| > \delta_\varepsilon L^\beta\right\}
\end{aligned}$$

and

$$\Pr\left\{\sqrt{(2l+1)}\left|\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - 1\right| > \delta_\varepsilon L^\beta\right\} \leq C\frac{\mathbb{E}\left[\sqrt{(2l+1)}\left|\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - 1\right|\right]^M}{\delta_\varepsilon^M L^{M\beta}} = O(L^{-M\beta}) \ ,$$

uniformly in l, see for instance [26], such that $M > 1/\beta$. Hence

$$\Pr\left\{\sup_l \sqrt{(2l+1)}\left|\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - 1\right| > \delta_\varepsilon L^\beta\right\} = O(L^{1-M\beta}) = o_L(1) \ .$$

For the general semiparametric case, the only difference is to be found in the expressions for $\mathbb{E}\widehat{C}_l$, which under Condition 2 becomes

$$\mathbb{E}\widehat{C}_l = G_0 l^{-\alpha_0}\left(1 + O\left(l^{-1}\right)\right),$$

where the bound $O\left(l^{-1}\right)$ is uniform over α by Assumption. As before, we hence obtain

$$\frac{\widehat{G}_k(\alpha)}{G_k(\alpha)} - 1$$

$$\begin{aligned}
&= \frac{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k \frac{\widehat{C}_l}{l^{-\alpha}} - \sum_{l=1}^{L}(2l+1)\left(\log l\right)^k \frac{G_0 l^{-\alpha_0}}{l^{-\alpha}}}{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k \frac{G_0 l^{-\alpha_0}}{l^{-\alpha}}} \\
&= \frac{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k G_0 l^{\alpha-\alpha_0}\left\{\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - \frac{\mathbb{E}\widehat{C}_l}{G_0 l^{-\alpha_0}} + \frac{\mathbb{E}\widehat{C}_l}{G_0 l^{-\alpha_0}} - 1\right\}}{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k G_0 l^{\alpha-\alpha_0}} \\
&= \frac{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k G_0 l^{\alpha-\alpha_0}\left\{\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - \frac{\mathbb{E}\widehat{C}_l}{G_0 l^{-\alpha_0}}\right\}}{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k G_0 l^{\alpha-\alpha_0}} \\
&\quad + \frac{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k G_0 l^{\alpha-\alpha_0}\left\{O(\frac{1}{l})\right\}}{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k G_0 l^{\alpha-\alpha_0}} \; .
\end{aligned}$$

The second summand is immediately observed to be $O(\frac{1}{L})$. By the same argument as before, for $0 < \beta < \frac{1}{2}$, we have, for all l :

$$\begin{aligned}
&\Pr\left\{\left|\frac{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k G_0 l^{\alpha-\alpha_0}\left\{\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - \frac{\mathbb{E}\widehat{C}_l}{G_0 l^{-\alpha_0}}\right\}}{\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k G_0 l^{\alpha-\alpha_0}}\right| > \delta_\varepsilon\right\} \\
\leq\ &\Pr\left\{L^\beta\left|\frac{\sum_l \sqrt{(2l+1)}\left(\log l\right)^k l^{\alpha-\alpha_0}}{\sum_l (2l+1)\left(\log l\right)^k l^{\alpha-\alpha_0}}\right| \frac{\sup_l \sqrt{(2l+1)}\left|\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - \frac{\mathbb{E}\widehat{C}_l}{G_0 l^{-\alpha_0}}\right|}{L^\beta} > \delta_\varepsilon\right\} \\
\leq\ &\Pr\left\{\sup_l \sqrt{(2l+1)}\frac{\mathbb{E}\widehat{C}_l}{G_0 l^{-\alpha_0}}\left|\frac{\widehat{C}_l}{\mathbb{E}\widehat{C}_l} - 1\right| > \delta_\varepsilon L^\beta\right\} \\
\leq\ &\Pr\left\{\sup_l \sqrt{(2l+1)}\left\{1 + O(\frac{1}{l})\right\}\left|\frac{\widehat{C}_l}{\mathbb{E}\widehat{C}_l} - 1\right| > \delta_\varepsilon L^\beta\right\} \; .
\end{aligned}$$

The rest of the proof is analogous to the argument we provided before, and hence omitted.

For the case where $\alpha_0 - \alpha \geq 2$, it suffices to note that

$$\frac{\widehat{G}_k(\alpha)}{G_k(\alpha)} = \frac{\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k \frac{\widehat{C}_l}{l^{-\alpha}}}{\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k \frac{G_0 l^{-\alpha_0}}{l^{-\alpha}}} > 0 \text{ , with probability } 1 \text{ ,}$$

and

$$\mathbb{E}\frac{\widehat{G}_k(\alpha)}{G_k(\alpha)} = \frac{\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k \frac{G_0 l^{-\alpha_0}}{l^{-\alpha}}\left\{1+O(\frac{1}{l})\right\}}{\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\left(\log l\right)^k \frac{G_0 l^{-\alpha_0}}{l^{-\alpha}}} = O(1) \ .$$

■

We are now in the position to establish the asymptotic behavior of $U_L(\alpha, \alpha_0)$ in (11), for which we have the following

Lemma 11 *For all* $2 > \alpha_0 - \alpha > \varepsilon > 0$, *we have that*

$$\lim_{L\to\infty}\left\{-\frac{(\alpha-\alpha_0)}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\log l + \log\frac{G(\alpha)}{G(\alpha_0)}\right\}$$

$$\begin{aligned}
&= \lim_{L\to\infty}\left[\log\left\{\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)l^{\alpha-\alpha_0}\right\} - \frac{(\alpha-\alpha_0)}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\log l\right] \\
&= (1+(\alpha-\alpha_0)/2) - \log(1+(\alpha-\alpha_0)/2) - 1 > \delta_\varepsilon > 0 \ .
\end{aligned}$$

Moreover, if $\alpha_0 - \alpha = 2$,

$$\lim_{L\to\infty}\frac{1}{\log\log L}\left\{-\frac{(\alpha-\alpha_0)}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\log l + \log\frac{G(\alpha)}{G(\alpha_0)}\right\} = 1 > 0 \ ,$$

and for $\alpha_0 - \alpha > 2$,

$$\lim_{L\to\infty}\frac{1}{\log L}\left\{-\frac{(\alpha-\alpha_0)}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\log l + \log\frac{G(\alpha)}{G(\alpha_0)}\right\} = \alpha_0 - \alpha - 2 > 0 \ .$$

Proof. Note first that

$$\lim_{L\to\infty}\frac{(\alpha-\alpha_0)}{\log L\times\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\log l = (\alpha-\alpha_0) \ , \tag{14}$$

and

$$\lim_{L\to\infty}\left\{\log L\right\}^{-1}\log\left\{\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)l^{\alpha-\alpha_0}\right\} = (\alpha-\alpha_0) \ , \tag{15}$$

with

$$\log\left\{\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)l^{\alpha-\alpha_0}\right\} - \frac{(\alpha-\alpha_0)}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\log l \geq 0$$

for all $L > 0$, because by Jensen's inequality

$$\log\mathbb{E}_{p_{l;L}}\left\{x^{\alpha-\alpha_0}\right\} - \mathbb{E}_{p_{l;L}}\log\left\{x^{\alpha-\alpha_0}\right\} > 0$$

where

$$p_{l;L} := \Pr\{x = l\} = \frac{2l+1}{\sum_{l=1}^{L}(2l+1)} \ .$$

To prove that the inequality is strict also in the limit, consider first the case $\alpha - \alpha_0 > -2$

$$\log\left\{\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)l^{\alpha-\alpha_0}\right\}$$

$$= \log\left\{\frac{(1+(\alpha-\alpha_0)/2)}{L^{\alpha-\alpha_0}\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)l^{\alpha-\alpha_0}\right\} - \log(1+(\alpha-\alpha_0)/2) + (\alpha-\alpha_0)\log L,$$

where

$$\begin{aligned}\frac{(1+(\alpha-\alpha_0)/2)}{L^{\alpha-\alpha_0}\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)l^{\alpha-\alpha_0} - 1 &= o_L(1) \ , \text{ whence} \\ \log\left\{\frac{(1+(\alpha-\alpha_0)/2)}{L^{\alpha-\alpha_0}\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)l^{\alpha-\alpha_0}\right\} &= o_L(1) \ .\end{aligned}$$

Thus

$$\log\left\{\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)l^{\alpha-\alpha_0}\right\} - \frac{(\alpha-\alpha_0)}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\log l$$

$$= -\log(1+(\alpha-\alpha_0)/2) + (\alpha-\alpha_0)\log L + - \frac{(\alpha-\alpha_0)}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\log l + o_L(1)$$

$$\begin{aligned}= \ & \frac{(\alpha-\alpha_0)}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)(\log L - \log l) \\ & - \frac{(\alpha-\alpha_0)}{2} + \frac{(\alpha-\alpha_0)}{2} - \log(1+(\alpha-\alpha_0)/2) + o_L(1) \ .\end{aligned}$$

Now

$$\begin{aligned}& \frac{(\alpha-\alpha_0)}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)(\log L - \log l) - \frac{(\alpha-\alpha_0)}{2} \\ = \ & \frac{2(\alpha-\alpha_0)}{L^2}\sum_{l=1}^{L} l \log\frac{L}{l} - \frac{(\alpha-\alpha_0)}{2} + o_L(1) \\ = \ & 2\,(\alpha-\alpha_0)\int_0^L \frac{x}{L}\log\frac{L}{x}\frac{dx}{L} - \frac{(\alpha-\alpha_0)}{2} + o_L(1) \\ = \ & -2(\alpha-\alpha_0)\int_0^1 x\log x dx - \frac{(\alpha-\alpha_0)}{2} + o_L(1) = o_L(1)\end{aligned}$$

because

$$\int_0^1 x\log x dx = \left[\frac{x^2}{2}\log x\right]_0^1 - \int_0^1 \frac{x^2}{2}\frac{1}{x}dx = -\frac{1}{4} \ .$$

We have hence proved that

$$\begin{aligned} & \log\left\{\frac{1}{\sum_{l=1}^L(2l+1)}\sum_{l=1}^L(2l+1)l^{\alpha-\alpha_0}\right\} - \frac{(\alpha-\alpha_0)}{\sum_{l=1}^L(2l+1)}\sum_{l=1}^L(2l+1)\log l \\ = & \ (1+(\alpha-\alpha_0)/2) - \log(1+(\alpha-\alpha_0)/2) - 1 + o_L(1) > 0 \ , \end{aligned}$$

for all $|\alpha-\alpha_0| > \varepsilon$, $\alpha-\alpha_0 > -2$.

Consider now the case $\alpha_0 - \alpha \geq 2$. We can rewrite:

$$-\frac{(\alpha-\alpha_0)}{\sum_{l=1}^L(2l+1)}\sum_{l=1}^L(2l+1)\log l + \log\frac{G(\alpha)}{G(\alpha_0)}$$

$$= \log\left\{\frac{1}{\sum_{l=1}^L(2l+1)}\sum_{l=1}^L(2l+1)l^{\alpha-\alpha_0}\right\} - \frac{(\alpha-\alpha_0)}{\sum_{l=1}^L(2l+1)}\sum_{l=1}^L(2l+1)\log l$$

$$= (\alpha_0-\alpha)\log L\left[\frac{\log\sum_{l=1}^L(2l+1)l^{-(\alpha_0-\alpha)}}{(\alpha_0-\alpha)\log L} - \frac{\log\sum_{l=1}^L(2l+1)}{(\alpha_0-\alpha)\log L} + \frac{1}{\sum_{l=1}^L(2l+1)}\sum_{l=1}^L(2l+1)\frac{\log l}{\log L}\right]$$

$$= (\alpha_0-\alpha)\log L\left[A_L + B_L + C_L\right] \ .$$

For the term A_L:

$$\sum_{l=1}^L(2l+1)l^{-(\alpha_0-\alpha)} = c\sum_{l=1}^L l^{1-(\alpha_0-\alpha)} + o_{L^{2-(\alpha_0-\alpha)}}(1) \to_L c > 1 \ ,$$

because $\sum_{l=1}^L l^{1-(\alpha_0-\alpha)}$ is a convergent series when the exponent $1-(\alpha_0-\alpha) < -1$; for $1-(\alpha_0-\alpha) = -1$, we have $\left\{\sum_{l=1}^L l^{1-(\alpha_0-\alpha)}/\log L\right\} \to 1$ and the argument is analogous. Therefore

$$(\alpha_0-\alpha)\log L \times [A_L] = \begin{cases} O(\log\log L) \text{ for } \alpha_0-\alpha = 2 \\ O(1) \text{ for } \alpha_0-\alpha > 2 \end{cases} .$$

As far as B_L is concerned, we have:

$$\begin{aligned} \log\sum_{l=1}^L(2l+1) &= \log L^2 + o(\log L) = \\ &= 2\log L + o(\log L), \end{aligned}$$

so that:

$$\lim_{L\to\infty} B_L = -\frac{2}{(\alpha_0-\alpha)};$$

finally, simple manipulations and standard properties of the logarithm (which is a slowly varying function, compare [5]) yield

$$\lim_{L\to\infty} C_L = \lim_{L\to\infty}\left[\frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)\frac{\log l}{\log L}\right] = 1 \ .$$

Summing up, we obtain:

$$\lim_{L\to\infty}\left\{(\alpha_0-\alpha)\log L\left[B_L+C_L\right]\right\} = \left\{\begin{array}{c} 0 \text{ for } \alpha_0-\alpha=2 \\ (\alpha_0-\alpha)-2>0 \text{ for } \alpha_0-\alpha>2 \end{array}\right. ,$$

and the claimed result follows. ■

In [34] a related computation was given for approximate Whittle estimates on stationary long memory processes in dimension $d=1$, i.e. the limiting lower bound turned out to be $(1+(\alpha-\alpha_0))-\log(1+(\alpha-\alpha_0))-1+o_L(1)>\delta_\varepsilon$. In view of this, we conjecture that for general d-dimensional spheres the lower bound will take the form

$$(1+\frac{(\alpha-\alpha_0)}{d})-\log(1+\frac{(\alpha-\alpha_0)}{d})-1+o_L(1)>\delta_\varepsilon \ .$$

Now we look at $T_L(\alpha,\alpha_0)$, for which we provide the following

Lemma 12 *Under Condition 2, as $L\to\infty$, we have*

$$\begin{aligned} \sup_\alpha \left|T_L(\alpha,\alpha_0)\right| &= o_p(1) \ , \textit{ for } \alpha_0-\alpha<2 \ , \\ \sup_\alpha \left|T_L(\alpha,\alpha_0)\right| &= O_p(1) \ , \textit{ for } \alpha_0-\alpha\geq 2 \ . \end{aligned}$$

Proof. For $\alpha_0-\alpha<2$, consider first

$$\frac{\widehat{G}(\alpha_0)}{G(\alpha_0)}-1 = \frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)(\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}}-1) \ ,$$

where we have easily, as $L\to\infty$,

$$\begin{aligned} \mathbb{E}\left\{\frac{\widehat{G}(\alpha_0)}{G(\alpha_0)}-1\right\} &= \frac{1}{\sum_{l=1}^{L}(2l+1)}\sum_{l=1}^{L}(2l+1)(\frac{G_0 l^{-\alpha_0}\left\{1+O(l^{-1})\right\}}{G_0 l^{-\alpha_0}}-1)\to 0 \ , \\ Var\left\{\frac{\widehat{G}(\alpha_0)}{G(\alpha_0)}\right\} &= \left\{\frac{1}{\sum_{l=1}^{L}(2l+1)}\right\}^2\sum_{l=1}^{L}(2l+1)^2 = O(\frac{1}{L}) \ , \end{aligned}$$

whence by Slutzky's lemma

$$\left\{\frac{\widehat{G}_L(\alpha_0)}{G_L(\alpha_0)}\xrightarrow{\mathbb{P}}1\right\}\Rightarrow\left\{\log\frac{\widehat{G}(\alpha_0)}{G(\alpha_0)}\xrightarrow{\mathbb{P}}0\right\} \ .$$

On the other hand, in view of Lemma 10, we have that:

$$\sup_\alpha\left|\log\frac{\widehat{G}(\alpha)}{G(\alpha)}\right| = o_p(1),$$

whence the result follows easily. The proof for $\alpha_0-\alpha\geq 2$ is immediate. ■

5 Asymptotic properties: The Central Limit Theorem

For the Central Limit Theorem, we shall exploit some classical argument on asymptotic Gaussianity for extremum estimates, as recalled for instance by [30].

Theorem 13 (Newey Mc Fadden CLT) *Assume that the extremum estimate $\widehat{\alpha}_N := \arg\min_{\alpha\in A} R(\alpha)$ satisfies the following conditions, as $N \to \infty$:*

1. $\widehat{\alpha}_N \xrightarrow{\mathbb{P}} \alpha_0$;
2. $\alpha_0 \notin \partial A$
3. $R_N(\alpha) \in C^2(I_\varepsilon(\alpha_0))$, *for all* N;
4. $R_N'(\alpha) = 0$ *a.s.;*
5. $\sqrt{N} R_N'(\alpha) \xrightarrow{d} \mathcal{N}(0,\Sigma)$;
6. *there exists* $F = F(\alpha)$, $F(\alpha_0) \neq 0$ *such that*

$$\sup_{\alpha:|\alpha-\alpha_0|<\varepsilon} \|R_N''(\alpha) - F(\alpha)\| \xrightarrow{\mathbb{P}} 0 \ ,$$

Then $\sqrt{N}(\widehat{\alpha} - \alpha_0) \xrightarrow{d} \mathcal{N}\left(0, (\Sigma/F(\alpha_0))^2\right)$

The proof is standard textbook material - here, we simply recall a few steps to fix some notation we shall use later. Conditions $2, 3, 4$ in this Theorem imply that $R_N'(\widehat{\alpha}) = 0$ a.s.. We can hence use a Taylor argument to expand $R_N'(\widehat{\alpha})$ around the point α_0:

$$0 = R_N'(\widehat{\alpha}) = R_N'(\alpha_0) + (\widehat{\alpha} - \alpha_0) R_N''(\alpha) + o\left((\widehat{\alpha} - \alpha_0)^2\right),$$

so that:

$$\sqrt{N}(\widehat{\alpha} - \alpha_0) = -\frac{\sqrt{N} R_N'(\alpha_0)}{R_N''(\overline{\alpha})},$$

where $\overline{\alpha}$ is a mean value that lies on the line segment between $\widehat{\alpha}$ and α_0. Because $\overline{\alpha} \xrightarrow{\mathbb{P}} \alpha_0$ and Condition 4, with probability approaching 1, we have

$$\begin{aligned} |R_N''(\overline{\alpha}) - F(\alpha_0)| &\leq |R_N''(\overline{\alpha}) - F(\overline{\alpha})| + |F(\overline{\alpha}) - F(\alpha_0)| \\ &\leq \sup_{\alpha\in A} |R_N''(\overline{\alpha}) - F(\overline{\alpha})| + |F(\overline{\alpha}) - F(\alpha_0)| \xrightarrow{\mathbb{P}} 0. \end{aligned}$$

Hence $F(\overline{\alpha}) \xrightarrow{\mathbb{P}} F(\alpha_0)$ and because $F(\alpha_0) \neq 0$, $F(\overline{\alpha})^{-1} \xrightarrow{\mathbb{P}} F(\alpha_0)^{-1}$ and the result follows easily by the Continuous Mapping Theorem.

We now introduce some auxiliary results whose proof is deferred to the Appendix.

Proposition 14 *Let*

$$Z_L(s) := \left[\sum_{l=1}^{L} (2l+1)\, l^{1+s} \sum_{l=1}^{L} (2l+1)\, l^{1+s} (\log l)^2 - \left(\sum_{l=1}^{L} (2l+1)\, l^{1+s} \log l \right)^2 \right] .$$

Then, for $s \in \mathbb{R}$*:*

$$\lim_{L\to\infty} \frac{1}{L^{4+2s}} Z_L(s) = \frac{1}{4\left(1+\frac{s}{2}\right)^4} . \tag{16}$$

Moreover, let $L_1 = L \cdot (1 - g(L))$*, where* $0 < g(L) < 1$ *is such that* $\lim_{L\to\infty} g(L) = 0$*. If*

$$Z_{L;g(L)}(s) = \sum_{l=L_1}^{L} (2l+1) l^{1+s} \sum_{l=L_1}^{L} (2l+1) l^{1+s} \left(\log^2 l\right) - \left(\sum_{l=L_1}^{L} (2l+1) l^{1+s} \log l \right)^2 , \tag{17}$$

we have

$$\lim_{L\to\infty} \frac{1}{L^{4\left(1+\frac{s}{2}\right)} g^4(L)} Z_{L;g(L)}(s) = K(s) \ , \tag{18}$$

where

$$K(s) = \frac{1}{\left(1+\frac{s}{2}\right)^2} \left(\frac{1}{12} s^2 - \frac{1}{8} s + \frac{1}{3} \right) .$$

Remark 15 *It is obvious that, if* $s = 0$*,*

$$K_0 = K(s)\big|_{s=0} = \frac{1}{3} . \tag{19}$$

Our purpose below is to show that the conditions provided in Theorem 13 in are satisfied by $\widehat{\alpha}_L$ and $\widehat{R}_L$ in the Spherical Whittle estimates, so that the following result will be established.

Theorem 16 *Let* $\widehat{\alpha}_L = \arg\min_{\alpha\in A} R_L(\alpha)$ *defined as in (10).*

a) Under Condition 2 we have that

$$\left\{ \mathbb{E} \left(\widehat{\alpha}_L - \alpha_0\right)^2 \right\}^{1/2} = O\left(\frac{\log L}{L}\right) \ , \text{ whence } \left(\widehat{\alpha}_L - \alpha_0\right) = O_p\left(\frac{\log L}{L}\right) \ , \text{ as } L \to \infty . \tag{20}$$

b) Under Condition 3 we have that

$$\frac{L}{4 \log L} \left(\widehat{\alpha}_L - \alpha_0\right) \longrightarrow_p -\kappa . \tag{21}$$

c) Under Condition 4 we have that

$$\frac{\sqrt{2}L}{4} \left(\widehat{\alpha}_L - \alpha_0\right) \xrightarrow{d} \mathcal{N}(0,1) . \tag{22}$$

Proof. We note first that under Condition 4, (22) is an immediate consequence of (21); on the other hand, the proof of (20) follows on exactly the same lines as (21), the only difference here being that the asymptotic bias term cannot be given an analytic expression but only bounded. It is then sufficient to establish (21), as we shall do below.

Following the notation introduced above, for each L there exist $\overline{\alpha}_L \in (\alpha_0 - \widehat{\alpha}, \alpha_0 + \widehat{\alpha})$ such that, with probability one:

$$(\widehat{\alpha}_L - \alpha_0) = -\frac{S_L(\alpha_0)}{Q_L(\overline{\alpha}_L)} \ ,$$

where $S_L(\alpha)$ is the score function corresponding to $R_L(\alpha)$, given by:

$$S_L(\alpha) = \frac{d}{d\alpha} R(\alpha) = \frac{\widehat{G}_1(\alpha)}{\widehat{G}(\alpha)} - \frac{1}{\sum_{l=1}^L (2l+1)} \sum_{l=1}^L (2l+1) \log l \ ,$$

and

$$Q_L(\alpha) = \frac{d}{d\alpha} S_L(\alpha) = \frac{d^2}{d\alpha^2} R(\alpha) = \frac{\widehat{G}_2(\alpha)\widehat{G}(\alpha) - \widehat{G}_1^2(\alpha)}{\widehat{G}^2(\alpha)}$$

$$= \frac{\sum_{l=1}^L (2l+1)(\log^2 l) \frac{\widehat{C}_l}{l^{-\alpha}} \left\{ \sum_{l=1}^L (2l+1) \frac{\widehat{C}_l}{l^{-\alpha}} \right\} - \left\{ \sum_{l=1}^L (2l+1)(\log l) \frac{\widehat{C}_l}{l^{-\alpha}} \right\}^2}{\left\{ \sum_{l=1}^L (2l+1) \frac{\widehat{C}_l}{l^{-\alpha}} \right\}^2} \ .$$

where $\widehat{G}(\alpha)$, $\widehat{G}_1(\alpha)$, $\widehat{G}_2(\alpha)$ are respectively the estimate of G and its first and second derivatives, as in Lemma 10. By direct substitution we have immediately:

$$S_L(\alpha) = \frac{1}{\sum_{l=1}^L (2l+1)} \sum_{l=1}^L (2l+1) \log l \left\{ \frac{\widehat{C}_l}{\widehat{G}(\alpha) l^{-\alpha}} - 1 \right\} \ .$$

Now,

$$\begin{aligned} S_L(\alpha_0) &= \frac{G_0}{\widehat{G}(\alpha_0)} \frac{1}{\sum_{l=1}^L (2l+1)} \sum_{l=1}^L (2l+1) \log l \left\{ \frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - \frac{\widehat{G}(\alpha_0)}{G_0} \right\} \\ &= \frac{G_0}{\widehat{G}(\alpha_0)} \overline{S}_L(\alpha_0) \ , \end{aligned}$$

where

$$\begin{aligned} \overline{S}_L(\alpha_0) &= \frac{1}{\sum_{l=1}^L (2l+1)} \sum_{l=1}^L (2l+1) \log l \left\{ \frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} - 1 \right\} \quad \text{and} \\ \frac{G_0}{\widehat{G}(\alpha_0)} &= 1 + o_p(1) \ , \text{ as } L \to \infty \ , \end{aligned}$$

in view of Lemma 10. Also

$$\begin{aligned}
\mathbb{E}\overline{S}_L(\alpha_0) &= \frac{1}{\sum_{l=1}^L (2l+1)} \sum_{l=1}^L (2l+1)\log l \left\{ \frac{C_l}{G_0 l^{-\alpha_0}} - 1 \right\} \\
&= \frac{\kappa}{\sum_{l=1}^L (2l+1)} \sum_{l=1}^L (2l+1) \frac{\log l}{l} + o(\frac{\log L}{L}) = O(\frac{\log L}{L}) \to 0 \ ,
\end{aligned}$$

and

$$\lim_{L\to\infty} 2L^2 Var\left\{ \overline{S}_L(\alpha_0) \right\} = 1 \tag{23}$$

In fact, we have:

$$Var\left\{ \overline{S}_L(\alpha_0) \right\} = V_1 + V_2 + V_3 \ ,$$

where

$$\begin{aligned}
V_1 &= \left\{ \frac{1}{\sum_{l=1}^L (2l+1)} \right\}^2 \sum_{l=1}^L (2l+1)^2 (\log l)^2 \, Var\left\{ \frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} \right\} \\
&= \left(\frac{1}{\sum_{l=1}^L (2l+1)} \right)^2 2 \sum_{l=1}^L (2l+1) (\log l)^2 \ ;
\end{aligned}$$

$$V_2 = \left\{ \frac{1}{\sum_{l=1}^L (2l+1)} \right\}^2 \left(\sum_{l=1}^L (2l+1)\log l \right)^2 Var\left(\frac{\widehat{G}(\alpha_0)}{G_0} \right) \ ;$$

$$V_3 = \frac{-2}{\sum_{l=1}^L (2l+1)} \sum_{l=1}^L (2l+1)\log l \, Cov\left(\frac{\widehat{C}_l}{C_l}, \frac{\widehat{G}(\alpha_0)}{G_0} \right) \cdot \frac{-2}{\sum_{l=1}^L (2l+1)} \sum_{l=1}^L (2l+1)\log l \ .$$

Now because

$$\begin{aligned}
Var\left(\frac{\widehat{G}(\alpha_0)}{G_0} \right) &= \left\{ \frac{1}{\sum_{l=1}^L (2l+1)} \right\}^2 \sum_{l=1}^L (2l+1)^2 \, Var\left\{ \frac{\widehat{C}_l}{G_0 l^{-\alpha_0}} \right\} \\
&= \frac{2}{\sum_{l=1}^L (2l+1)} \ ; \qquad (24)
\end{aligned}$$

$$\begin{aligned}
Cov\left(\frac{\widehat{C}_l}{C_l}, \frac{\widehat{G}(\alpha_0)}{G_0} \right) &= \frac{1}{\sum_{l'=1}^L (2l'+1)} \sum_{l'=1}^L (2l'+1) \, Cov\left(\frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_{l'}}{C_{l'}} \right) \\
&= \frac{2}{\sum_{l'=1}^L (2l+1)} \ ; \qquad (25)
\end{aligned}$$

we have

$$Var\left\{ \overline{S}_L(\alpha_0) \right\}$$

$$
\begin{aligned}
&= \frac{2}{\left(\sum_{l=1}^{L}(2l+1)\right)^{3}} \left(\sum_{l=1}^{L}(2l+1) \sum_{l=1}^{L}(2l+1)(\log l)^{2} - \left(\sum_{l=1}^{L}(2l+1)\log l \right)^{2} \right) \\
&= \frac{2}{L^{6}} \frac{L^{4}}{4} = \frac{1}{2L^{2}} \ ,
\end{aligned}
$$

by using (16) and (30) with $s = 0$ to obtain (23). In order to establish the Central Limit Theorem, it is sufficient to perform a careful analysis of fourth-order cumulants (note our statistics belong to the second-order Wiener chaos with respect to a Gaussian white noise random measure). Write:

$$
LS_L(\alpha_0) = \frac{1}{L + O_L(1)} \sum_{l} (A_l + B_l) \ ,
$$

where

$$
\begin{aligned}
A_l &= (2l+1)\log l \left\{ \frac{\widehat{C}_l}{C_l} - 1 \right\} && (26) \\
B_l &= (2l+1)\log l \left\{ \frac{\widehat{G}_L(\alpha_0)}{G_0} - 1 \right\} \ . && (27)
\end{aligned}
$$

In the Appendix, we show that

$$
\frac{1}{L^4} cum \left\{ \sum_{l_1}(A_{l_1} + B_{l_1}), \sum_{l_2}(A_{l_2} + B_{l_2}), \sum_{l_3}(A_{l_3} + B_{l_3}), \sum_{l_4}(A_{l_4} + B_{l_4}) \right\} = O_L(\frac{\log^4 L}{L^2}) \ .
$$

whence the Central Limit theorem follows easily from results in [31]. Indeed, using recent results from the latter authors a stronger result follows, that is

$$
d_{TV}(\sum_{l=1}^{L} X_{l;L}, Z) = O(\frac{1}{L}) \ , \ Z \overset{d}{=} \mathcal{N}(0,1) \ ,
$$

where $d_{TV}(W,V)$ denotes the total variation distance between the random variables W, V, i.e.

$$
d_{TV}(W,V) = \sup_{x} \left| \Pr\{W \in B\} - \Pr\{V \in B\} \right| \ , \text{ any Borel set } B \ .
$$

Also

$$
\begin{aligned}
\frac{L}{\log L} \mathbb{E}\overline{S}_L(\alpha_0) &= \kappa \frac{L}{\sum_{l=1}^{L}(2l+1)} \sum_{l=1}^{L} \frac{(2l+1)}{l} \frac{\log l}{\log L} + o(1) \\
&\to -\kappa \ , \text{ as } L \to \infty \ .
\end{aligned}
$$

Let us now focus on the second order derivative. From consistency, it is sufficient to focus on $|\alpha - \alpha_0| < 2$; here we can apply again Lemma 10, replacing the

random quantities $\widehat{G}_k(\alpha)$ with the corresponding deterministic $G_k(\alpha)$ values, to obtain

$$Q_L(\alpha) = \frac{G_2(\alpha)G(\alpha) - G_1^2(\alpha)}{G^2(\alpha)} + o_p(1) \ ,$$

uniformly over α. It is convenient to write

$$\frac{G_2(\alpha)G(\alpha) - G_1^2(\alpha)}{G^2(\alpha)} = \frac{Q_L^{num}(\alpha)}{Q_L^{den}(\alpha)} \ .$$

Let us start by studying $Q_L^{den}(\alpha)$. We have, by using (30) with $s = 0$ and $s = \alpha - \alpha_0$:

$$\begin{aligned}
\frac{Q_L^{den}(\alpha)}{L^{2(\alpha-\alpha_0)}} &= \frac{1}{L^{2(\alpha-\alpha_0)}} \left(\frac{1}{\sum_{l=1}^{L}(2l+1)} \sum_{l=1}^{L} (2l+1) \frac{G_0 l^{-\alpha_0}}{l^{-\alpha}} \right)^2 \\
&= G_0^2 \left(\frac{1}{\left(1 + \frac{\alpha-\alpha_0}{2}\right)^2} + o_L(1) \right).
\end{aligned}$$

Consider now $Q_L^{num}(\alpha)$, where we have:

$$\frac{Q_L^{num}(\alpha)}{L^{2(\alpha-\alpha_0)}} =$$

$$\left(\frac{G_0 L^{-(\alpha-\alpha_0)}}{\sum_{l=1}^{L}(2l+1)} \right)^2 \left[\left(\sum_{l=1}^{L} (2l+1) \frac{l^{-\alpha_0}}{l^{-\alpha}} \log^2 l \right) \left(\sum_{l=1}^{L} (2l+1) \frac{l^{-\alpha_0}}{l^{-\alpha}} \right) - \left(\sum_{l=1}^{L} (2l+1) \frac{l^{-\alpha_0}}{l^{-\alpha}} \log l \right)^2 \right]$$

$$= \frac{G_0^2}{L^{4+2(\alpha-\alpha_0)}} \left[\left(\sum_{l=1}^{L} (2l+1) \frac{l^{-\alpha_0}}{l^{-\alpha}} \log^2 l \right) \left(\sum_{l=1}^{L} (2l+1) \frac{l^{-\alpha_0}}{l^{-\alpha}} \right) - \left(\sum_{l=1}^{L} (2l+1) \frac{l^{-\alpha_0}}{l^{-\alpha}} \log l \right)^2 \right]$$

$$= G_0^2 \left[\frac{1}{4\left(1 + \frac{\alpha-\alpha_0}{2}\right)^4} \right] + o_L(1)$$

by using (16), $s = \alpha - \alpha_0$. Combining all terms, we find that, uniformly over α

$$Q_L(\alpha) = \frac{G_0^2 \frac{1}{4\left(1+\frac{\alpha-\alpha_0}{2}\right)^4} + o_L(1)}{G_0^2 \left(\frac{1}{\left(1+\frac{\alpha-\alpha_0}{2}\right)^2} + o_L(1) \right)} = \frac{1}{4\left(1 + \frac{\alpha-\alpha_0}{2}\right)^2} + o_L(1) \ .$$

Finally, from the consistency result

$$\left(1 + \frac{\overline{\alpha}_L - \alpha_0}{2}\right)^2 \xrightarrow{\mathbb{P}} 1 \ , \ Q_L(\overline{\alpha}_L) \xrightarrow{\mathbb{P}} \frac{1}{4} \ .$$

Thus by Slutzky Theorem we have:

$$\frac{\sqrt{2}L}{4} \frac{S_L(\alpha_0)}{Q_L(\overline{\alpha}_L)} \xrightarrow{d} \mathcal{N}\left(-\sqrt{2}\kappa, 1\right) \ ,$$

as claimed. ■

In the Appendix we describe in details the result concerning the analysis of fourth-order cumulants.

Remark 17 *In the statement of the previous Theorem, we decided to report normalization factors in the neatest possible form. A careful inspection of the proofs reveals however that the asymptotic result in (21) and (22) can be improved in finite samples introducing a correction factor* $c_L = \frac{1}{L}\sum_{l=1}^{L} \frac{\log l}{\log L} \to 1$, *as* $L \to \infty$, *as follows*

$$\frac{L}{4\log L \times c_L}\left(\widehat{\alpha}_L - \alpha_0\right) \longrightarrow_p \kappa$$

under condition (3), and

$$\frac{\sqrt{2}L}{4 \times c_L}\left(\widehat{\alpha}_L - \alpha_0\right) \xrightarrow{d} \mathcal{N}\left(0,1\right) \ ,$$

under condition (4). Note that $c_L < 1$ *for all finite* L, *whence the asymptotic bias and variance are slightly underestimated in Theorem 16. For instance, the correction factors for* $L = 1000, 2000, 4000$ *are, respectively,* $c_{1000} \simeq 0.86$, $c_{2000} \simeq 0.87$, *and* $c_{4000} \simeq 0.88$.

The previous result provides a sharp rate of convergence for the spherical Whittle estimator. However in the general case the asymptotic bias term $-\sqrt{2}\kappa$ is unknown, which makes inference unfeasible. To address these issues, we shall consider in the next Section an alternative *narrow-band* estimator (compare [34]) which achieves an unbiased limiting distribution, to the price of a log factor in the rate of convergence.

6 Narrow-band estimates

In the previous Section, we have shown that under Conditions 2, 3, it is possible to establish a rate of convergence for the spherical Whittle estimates; however, due to the presence of an asymptotic bias term, statistical inference turned out to be unfeasible. The purpose of this Section is to propose a narrow band estimator allowing for feasible inference under broad circumstances. We start from the following

Definition 18 *The* Narrow-Band Spherical Whittle estimator *for the parameters* $\vartheta = (\alpha, G)$ *is provided by*

$$(\widehat{\alpha}_{L;L_1}, \widehat{G}_{L;L_1}) := \arg\min_{\alpha,G} \sum_{l=L_1}^{L} \left\{ (2l+1)\frac{\widehat{C}_l}{Gl^{-\alpha}} - (2l+1)\log\frac{\widehat{C}_l}{Gl^{-\alpha}} \right\} \ ,$$

or equivalently

$$\begin{aligned} \widehat{\alpha}_{L;L_1} &= \arg\min_{\alpha} R_{L;L_1}(\alpha, \widehat{G}(\alpha)), \qquad (28) \\ R_{L;L_1}(\alpha, \widehat{G}) &= \left(\log \widehat{G}_{L;L_1}(\alpha) - \frac{\alpha}{\sum_{l=L_1}^{L}(2l+1)} \sum_{l=L_1}^{L} (2l+1)\log l \right) , \end{aligned}$$

where $L_1 < L$ is chosen such that

$$L - L_1 \to \infty \ , \ \frac{L}{L_1} = 1 + O(\frac{1}{\log L}) \ , \ as\ L \to \infty \ .$$

For definiteness, we shall take

$$L_1 = L\,(1 - g\,(L)) \ \text{, where } g\,(L) = \frac{1}{\log L} \ .$$

pretending for notational simplicity that L_1 so defined is an integer (if this isn't the case, modified arguments taking integer parts are completely trivial).

Theorem 19 *Let $\widehat{\alpha}_{L;L_1}$ defined as in (28). Then under Condition 3 we have*

$$\frac{L \cdot g\,(L)^{\frac{3}{2}}}{\sqrt{12}} \left(\widehat{\alpha}_{L;L_1} - \alpha_0\right) \xrightarrow{d} \mathcal{N}\,(0,1) \ .$$

Proof. The proof of the consistency for $\widehat{\alpha}_{L;L_1}$ can be carried out analogously to the argument provided in Section 4, and hence is omitted for brevity's sake. The proof for the Central Limit Theorem can also be carried along the same lines as done earlier, noting in particular that for the form (6) of C_l

$$\begin{aligned} \mathbb{E}\overline{S}_{L;L_1}(\alpha_0) &= \frac{1}{\sum_{l=L_1}^{L}(2l+1)} \sum_{l=L_1}^{L} (2l+1)\{\log l\} \left\{ \frac{C_l}{G_0 l^{-\alpha_0}} - \frac{\widehat{G}_{L;L_1}}{G_0} \right\} \\ &= \frac{\kappa}{\sum_{l=L_1}^{L}(2l+1)} \sum_{l=L_1}^{L} \left[(2l+1)\frac{\log l}{l} - \frac{\sum_{l=L_1}^{L}(2+\frac{1}{l})}{\sum_{l=L_1}^{L}(2l+1)} \right] \\ &= \kappa \frac{\log L_1}{L_1} + o\left(\frac{\log L_1}{L_1}\right) = O(\frac{\log L_1}{L_1}) \ , \end{aligned}$$

and

$$\begin{aligned} & L \cdot g\,(L)^{\frac{3}{2}}\, \mathbb{E}\left[\overline{S}_{L;L_1}(\alpha_0)\right] \\ &= O\left(\frac{\log L_1}{L_1}\right) \frac{L}{\log^{\frac{3}{2}} L} \\ &= O(\frac{L}{L_1}) O\left(\frac{\log L}{\log^{\frac{3}{2}} L}\right) = O\left(\frac{1}{\log^{\frac{1}{2}} L}\right) = o_L(1) \ . \end{aligned}$$

On the other hand

$$Var\left\{\overline{S}_{L;L_1}(\alpha_0)\right\}$$

$$\begin{aligned}
&= \frac{1}{\left[\sum_{l=L_1}^{L}(2l+1)\right]^2} Var\left\{\sum_{l=L_1}^{L}(2l+1)\{\log l\}\left(\frac{\widehat{C}_l}{G_0 l^{-\alpha_0}}-\frac{\widehat{G}_{L;L_1}(\alpha)}{G_0}\right)\right\} \\
&= \frac{2}{\left[\sum_{l=L_1}^{L}(2l+1)\right]^3}\left(\sum_{l=L_1}^{L}(2l+1)\sum_{l=L_1}^{L}(2l+1)\left\{\log^2 l\right\}-\left(\sum_{l=L_1}^{L}(2l+1)\{\log l\}\right)^2\right) = \\
&= \frac{2}{\left[\sum_{l=L_1}^{L}(2l+1)\right]^3} Z_{L;g(L)}(0) \ . \qquad (29)
\end{aligned}$$

by using (24) and (25) and following the notation of Proposition 14 with $s=0$.

Proposition 14 leads to:

$$\frac{1}{4}Z_{L;g(L)} = \frac{1}{3}g^4(L)L^4 + o\left(g^4(L)L^4\right) \ ,$$

while

$$\left[\sum_{l=L_1}^{L}(2l+1)\right]^3 = \left(L^2-L_1^2\right)^3 = 8L^6\left(g(L)-\frac{1}{2}g^2(L)\right)^3 = 8L^6 g^3(L) + o_L\left(L^6 g^3(L)\right) \ .$$

By substituting these results in (29) we obtain

$$Var\left\{\overline{S}_{L;L_1}(\alpha_0)\right\} = \frac{g(L)}{12L^2} = \frac{1}{12L^2\log(L)} \ .$$

Rewrite now the term $Q_{L_1L}(\alpha)$ as

$$Q_{L_1;L}(\alpha) = \frac{Q^{num}_{L_1;L}(\alpha)}{Q^{den}_{L_1;L}(\alpha)} \ .$$

where we have:\

$$Q^{num}_{L_1;L}(\alpha) = \frac{G_0^2}{\left(\sum_{l=1}^{L}(2l+1)\right)^2} Z_{L,g(L)}(s) \,,$$

$$Q^{den}_{L_1;L}(\alpha) = G_0^2\left(\frac{1}{\sum_{l=L_1}^{L}(2l+1)}\right)^2\left(\sum_{l=L_1}^{L}(2l+1)l^s\right)^2 .$$

where $s=\alpha-\alpha_0$.

From (30), we have

$$\begin{aligned} Q^{den}_{L_1;L}(\alpha) &= G_0^2 \frac{L^{4\left(1+\frac{s}{2}\right)}\left(1-(1-g(L))^{2\left(1+\frac{s}{2}\right)}\right)^2}{L^4\left(1-(1-g(L))^2\right)^2} + o_L(1) \\ &= \frac{G_0^2 L^{2s}}{\left(1+\frac{s}{2}\right)^2} \frac{\left(1-(1-g(L))^{2\left(1+\frac{s}{2}\right)}\right)^2}{\left(1-(1-g(L))^2\right)^2} + o_L(1) \\ &= \frac{4G_0^2 L^{2s} g^2(L)}{\left(1-(1-g(L))^2\right)^2} . + o_L(1) \end{aligned}$$

Consider now $Q^{num}_{L_1;L}(\alpha)$, where we have:

$$Q^{num}_{L_1;L}(\alpha) = G_0^2 \frac{L^{2s} g^4(L) K(s)}{\left(1-(1-g(L))^2\right)^2} + o_L(1) \ .$$

Combining the two results, we obtain:

$$\lim_{L\to\infty} Q_{L_1;L}(\alpha) = \frac{g^2(L) K(s)}{4} \ .$$

Finally, from the consistency results, we have:

$$\frac{12}{g^2(L)} Q_{L_1;L}(\overline{\alpha}) \to_p 1 \ .$$

The analysis of fourth-order moments is exactly the same as in the previous Section, and the result follows accordingly. ■

Remark 20 *It should be noted that an asymptotic unbiased estimator is obtained with the loss of only a logarithmic term to the power* $3/2$ *in the rate of convergence.*

7 Numerical Results

In this Section we present some numerical evidence to support the asymptotic results provided earlier. More precisely, using the statistical software R, for given fixed values of L, α_0 and G_0 and the alternative conditions discussed in the previous Section, we sample random values for the angular power spectra $\widehat{C}_l$ and we implement standard and narrow-band estimates. We start by analyzing the simplest model, i.e. the one corresponding to Condition 4. Here we fixed $G_0 = 2$.In Figure 1 we report the distribution of $\widehat{\alpha}_L - \alpha_0$ normalized by a factor $\sqrt{2}L/4$. In Table 1, we report instead the sample frequencies corresponding to the quantiles $q = 0.05, 0.25, 0.50.0.75, 0.95$ for a $\mathcal{N}(0,1)$ distribution.

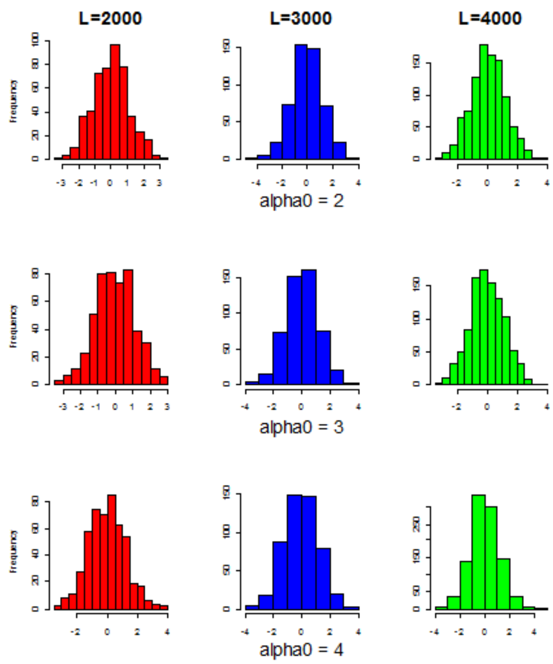

Figure 1: Distribution of normalized $(\widehat{\alpha}_L - \alpha_0)$ by varying L and α_0, under Condition 4.

			Sample Frequencies						
		L	-1.96	-1	-0.68	0	0.68	1	1.96
α_0	2	2000	4	19.2	29.2	48.6	22.8	14.2	4
		3000	4.5	18.4	26.8	51	23.33	14.36	3.6
		4000	4.4	17.7	25.2	49.1	23.43	13.87	4.8
	3	2000	4.4	19.2	29.2	51.5	24.03	15.17	3.6
		3000	4.3	18.4	26.8	48.9	23.2	13.43	3.8
		4000	4.2	17.9	26.4	50.8	23.07	14.13	3.7
	4	2000	4.4	21.6	30.2	50.9	22.73	14.94	5.5
		3000	4.2	21.2	29.8	50.4	25.07	15.87	4.3
		4000	4.2	17.9	27.1	50.4	22.7	13.73	4.2

Table 1: Quantiles of $L/4\sqrt{2}\log L\,(\widehat{\alpha}_L - \alpha_0)$

Table 2 provides the results for the classical Shapiro-Wilk Gaussianity Test performed on simulations obtained by varying α_0 and the number of multipoles L. Asymptotic Gaussianity is clearly supported.

			Shapiro-Wilk Test	
		L	W	$p-value$
α_0	2	2000	0.9976	0.685
		3000	0.9978	0.667
		4000	0.9983	0.373
	3	2000	0.9976	0.691
		3000	0.9980	0.842
		4000	0.9985	0.945
	4	2000	0.9987	0.670
		3000	0.998	0.286
		4000	0.9985	0.578

Table 2: Shapiro-Wilk test under Condition 4.

Let us now focus on the more general Condition 3. Figure 2 represents the empirical distribution of $\left(L/4\sqrt{2}\log L\right)(\widehat{\alpha}_L - \alpha_0)$ in case $\alpha_0 = 3$, $\kappa = 1$ and the corresponding narrow-band estimates, whose results are summarized in Table 3. The improvement in the bias factor with the latter procedure is immediately evident.

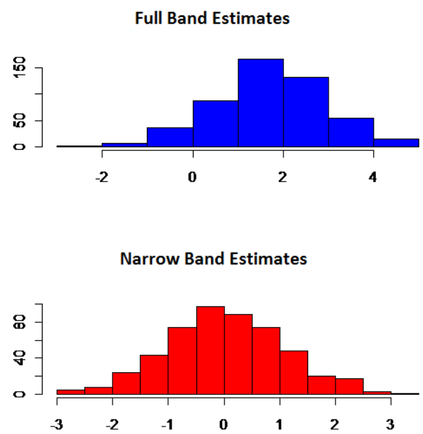

Figure 2: Comparison among biased and narrow estimates ($\kappa = 1$, $L = 2000$, $\alpha_0 = 3$), under Condition 3.

				Shapiro-Wilk Test	
L	L_1	mean	var	W	$p - value$
2000	1550	0.072	0.959	0.9985	0.950
	1700	0.018	0.951	0.997	0.495
	1850	-0.016	1.004	0.9977	0.739
3000	2400	0.092	1.130	0.9949	0.920
	2600	0.072	0.928	0.9951	0.745
	2800	-0.02	1.06	0.9965	0.340
4000	3250	0.006	0.985	0.9968	0.443
	3500	0.004	1.097	0.998	0.834
	3750	0.0007	1.073	0.9982	0.874

Table 4: Normalized Narrow bands data, under Condition 3, $\kappa = 1$, $\alpha_0 = 4$, $G_0 = 2$.

Considering the correction term c_L from Remark 17, the sample bias is consistent with the asymptotic value to three decimal digits.

Finally, in Figure 3, we report the results obtained on a set of simulations under Condition 2, where we have:

$$G(l) = G_0 \left\{ 1 + \frac{1}{l} - \frac{1}{l^2} \right\} ,$$

with $G_0 = 2$, $\alpha_0 = 4$, $L = 4000$, $L_1 = 3750$.

We obtain a mean value $\mathbb{E}\left(\widehat{\alpha}_L - \alpha_0\right) = 0.040$ and a normalized variance of 0.9918. Shapiro-Wilk Gaussianity test gives as result $W = 0.9981$ with a $p - value = 0.8669$.

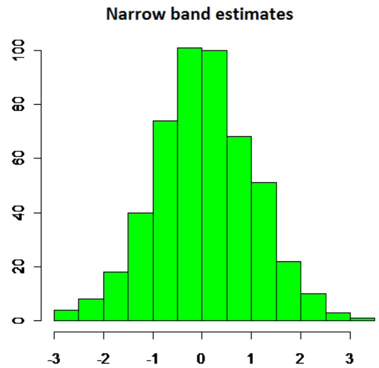

Figure 3: Distribution of normalized $(\widehat{\alpha}_L - \alpha_0)$ under Condition 2.

8 Conclusions

We view this paper as a first contribution in area which deserves much further research, i.e. the investigation of asymptotic properties for parametric estimators on a single realization of an isotropic random field on the sphere. As mentioned earlier, an enormous amount of applied papers have focussed on this issue, especially in a Cosmological framework, but no rigorous results seem currently available. Our results suggest that consistency and asymptotic Gaussianity are feasible for spectral index estimators, the rate of convergence being $L/\log L$; these estimates are centred on zero in "parametric" circumstances, i.e. where the correct model being provided for C_l up to a factor $o(\frac{1}{l})$; when the latter assumption fails, alternatively, narrow-band estimates can be entertained; these estimates ensure convergence to a zero-mean Gaussian distribution, convergence rate becoming $L/\log^2 L$.

Many questions are left open by these preliminary results. The first we mention is the characterization of a whole class of parameters for which asymptotic Gaussianity and consistency may continue to hold. More challenging is the possibility to relax the Gaussian assumption and consider more general, finite-variance isotropic Gaussian fields. In this respect, results in [25] suggest that the Gaussianity assumption may indeed play a crucial role, as high-frequency consistency and Gaussianity seem very tightly related, for instance when considering the asymptotic behavior of the angular power spectrum. It seems also important to explore the connection between the spherical estimates we have been considering and fixed-domain asymptotic results for Matern-type covariances which have been discussed on $\mathbb{R}^d$ by [23], [1], [36] and others.

More interestingly, we aim at relaxing some of the assumptions introduced in this paper to make this work more directly applicable on existing datasets. The harmonic estimates we have been focussing on hence require the observation of the random field on the full sphere. This condition often fails in practice: for

instance, in a Cosmological framework large regions of the sky are not masked by Foreground sources such as the Milky Way. In ongoing research (see [10]), we are hence considering a Whittle-type estimator based on spherical wavelets (needlets, see [29], [27], [3]), rather than standard Fourier analysis. We believe that the material in the present paper can provide also the right starting point for these kind of further developments.

A Appendix

We start with the proof of Proposition 14. For the sake of brevity, we prove only (18), omitting the proof of (16), considering it a particular case in which $L_1 = 1$. The following Lemma is pivotal for the proof.

Lemma 21 *Let $L_1 < L$, then we have*

$$\int_{L_1}^{L} 2x^{1+s}dx = \frac{1}{\left(1+\frac{s}{2}\right)}\left(L^{2\left(1+\frac{s}{2}\right)} - L_1^{2\left(1+\frac{s}{2}\right)}\right) \ ; \tag{30}$$

$$\int_{L_1}^{L} 2x^{1+s}\log x dx = -\frac{L^{2\left(1+\frac{s}{2}\right)} - L_1^{2\left(1+\frac{s}{2}\right)}}{2\left(1+\frac{s}{2}\right)^2} + \frac{L^{2\left(1+\frac{s}{2}\right)}\log L - L_1^{2\left(1+\frac{s}{2}\right)}\log L_1}{\left(1+\frac{s}{2}\right)} \ ; \tag{31}$$

$$\begin{aligned}\int_{L_1}^{L} 2x^{1+s}\log^2 x dx &= \frac{L^{2\left(1+\frac{s}{2}\right)} - L_1^{2\left(1+\frac{s}{2}\right)}}{2\left(1+\frac{s}{2}\right)^3} - \frac{L^{2\left(1+\frac{s}{2}\right)}\log L - L_1^{2\left(1+\frac{s}{2}\right)}\log L_1}{\left(1+\frac{s}{2}\right)^2} \\ &\quad + \frac{L^{2\left(1+\frac{s}{2}\right)}\log^2 L - L_1^{2\left(1+\frac{s}{2}\right)}\log^2 L_1}{\left(1+\frac{s}{2}\right)} \ .\end{aligned} \tag{32}$$

Proof. First of all, observe that for $u \geq 0$

$$\sum_{l=L_1}^{L}(2l+1)l^s\left(\log l\right)^u = \int_{L_1}^{L} 2x^{1+s}\log^u x dx + o_L(1) \ .$$

Simple calculations lead to:

$$\int_{L_1}^{L} 2x^{1+s}dx = \left.\frac{x^{2\left(1+\frac{s}{2}\right)}}{\left(1+\frac{s}{2}\right)}\right|_{x=L_1}^{L} = \frac{1}{\left(1+\frac{s}{2}\right)}\left(L^{2\left(1+\frac{s}{2}\right)} - L_1^{2\left(1+\frac{s}{2}\right)}\right) \ ;$$

By the same procedure we obtain (31) and (32). ■

Proof. (Proposition 14) We start by observing that

$$\left(\sum_{l=L_1}^{L}(2l+1)l^s\log^2 l\right)\left(\sum_{l=L_1}^{L}(2l+1)l^s\right)$$

$$
\begin{aligned}
&= \frac{\left(L^{2\left(1+\frac{s}{2}\right)}-L_1^{2\left(1+\frac{s}{2}\right)}\right)^2}{\left(1+\frac{s}{2}\right)^2}\left(\frac{1}{2\left(1+\frac{s}{2}\right)^2}+\frac{\log L}{\left(1+\frac{s}{2}\right)}+\log^2 L\right) \\
&\quad +\frac{\left(L^{2\left(1+\frac{s}{2}\right)}-L_1^{2\left(1+\frac{s}{2}\right)}\right)}{\left(1+\frac{s}{2}\right)^3}L_1^{2\left(1+\frac{s}{2}\right)}\log\left(1-g\left(L\right)\right)\times \\
&\quad \times\left(\frac{1}{\left(1+\frac{s}{2}\right)}-2\log L-\log^2\left(1-g\left(L\right)\right)\right)+o_L\left(1\right) \ ;
\end{aligned}
$$

$$
\left(\sum_{l=L_1}^{L}(2l+1)l^s\log l\right)^2
$$

$$
\begin{aligned}
&= \frac{\left(L^{2\left(1+\frac{s}{2}\right)}-L_1^{2\left(1+\frac{s}{2}\right)}\right)^2}{\left(1+\frac{s}{2}\right)^2}\left(\frac{1}{4\left(1+\frac{s}{2}\right)^2}-\frac{\log L}{\left(1+\frac{s}{2}\right)}+\log^2 L\right) \\
&\quad +\frac{\left(L^{2\left(1+\frac{s}{2}\right)}-L_1^{2\left(1+\frac{s}{2}\right)}\right)}{\left(1+\frac{s}{2}\right)^2}L_1^{2\left(1+\frac{s}{2}\right)}\log\left(1-g\left(L\right)\right)\left(\frac{1}{\left(1+\frac{s}{2}\right)}-2\log L\right) \\
&\quad +\frac{L_1^{4\left(1+\frac{s}{2}\right)}\log^2\left(1-g\left(L\right)\right)}{\left(1+\frac{s}{2}\right)^2}+o_L\left(1\right) \ ,
\end{aligned}
$$

so we obtain

$$
\begin{aligned}
Z_{L,g(L)}\left(s\right) &= \frac{\left(L^{2\left(1+\frac{s}{2}\right)}-L_1^{2\left(1+\frac{s}{2}\right)}\right)^2}{4\left(1+\frac{s}{2}\right)^4}-\frac{L^{2\left(1+\frac{s}{2}\right)}L_1^{2\left(1+\frac{s}{2}\right)}\log^2\left(1-g\left(L\right)\right)}{\left(1+\frac{s}{2}\right)^2}+o_L\left(1\right) \\
&= \frac{L^{4\left(1+\frac{s}{2}\right)}\left(\left(1-\left(1-g\left(L\right)\right)^{2\left(1+\frac{s}{2}\right)}\right)\right)^2}{4\left(1+\frac{s}{2}\right)^4} \\
&\quad -\frac{L^{4\left(1+\frac{s}{2}\right)}\left(1-g\left(L\right)\right)^{2\left(1+\frac{s}{2}\right)}\log^2\left(1-g\left(L\right)\right)}{\left(1+\frac{s}{2}\right)^2}+o_L\left(1\right).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\log^2(1-g\left(L\right)) &= \left(-g\left(L\right)-\frac{1}{2}g^2\left(L\right)-\frac{1}{3}g^3\left(L\right)+O\left(g^4\left(L\right)\right)\right)^2 \\
&= g^2\left(L\right)+g^3\left(L\right)+\left(\frac{11}{12}\right)g^4\left(L\right)+o\left(g^4\left(L\right)\right) \ ,
\end{aligned}
$$

while

$$\frac{(1-g(L))^{2\left(1+\frac{s}{2}\right)}}{\left(1+\frac{s}{2}\right)} = \frac{1}{\left(1+\frac{s}{2}\right)} - 2g(L) + \left(2\left(1+\frac{s}{2}\right)-1\right)g^2(L) - \frac{\left(2\left(1+\frac{s}{2}\right)-1\right)\left(2\left(1+\frac{s}{2}\right)-2\right)}{3}g^3(L) + o\left(g^3(L)\right) \ .$$

Thus

$$\frac{L^{4\left(1+\frac{s}{2}\right)}\left(\left(1-(1-g(L))^{2\left(1+\frac{s}{2}\right)}\right)\right)^2}{4\left(1+\frac{s}{2}\right)^4}$$

$$= \frac{L^{4\left(1+\frac{s}{2}\right)}g^2(L)}{\left(1+\frac{s}{2}\right)^2}[1+(s+1)g(L)+ +\frac{1}{4}(s+1)\left(\frac{7}{3}s+1\right)g^2(L)] + o\left(L^4g^4(L)\right) \ ,$$

while simple calculations lead to

$$\frac{L^{4\left(1+\frac{s}{2}\right)}(1-g(L))^{2\left(1+\frac{s}{2}\right)}\log^2(1-g(L))}{\left(1+\frac{s}{2}\right)^2}$$

$$= \frac{L^{4\left(1+\frac{s}{2}\right)}g^2(L)}{\left(1+\frac{s}{2}\right)^2}\left(1+(s+1)g(L) + \left(\frac{s^2}{2}+\frac{23}{24}s-\frac{1}{12}\right)g^2(L)\right) + \ o\left(L^4g^4(L)\right) \ .$$

By using (18), we have

$$Z_{L,g(L)}(s) = \frac{L^{4\left(1+\frac{s}{2}\right)}g^4(L)}{\left(1+\frac{s}{2}\right)^2}K(s) + \ o\left(L^4g^4(L)\right) \ ,$$

as we claimed. ■

Now we present the result about the analysis of the fourth-order cumulants.

Lemma 22 *Let A_l and B_l be defined as in 26 and 27. As $L \to \infty$,*

$$\frac{1}{L^4}cum\left\{\sum_{l_1}(A_{l_1}+B_{l_1}),\sum_{l_2}(A_{l_2}+B_{l_2}),\sum_{l_3}(A_{l_3}+B_{l_3}),\sum_{l_4}(A_{l_4}+B_{l_4})\right\} = O_L\left(\frac{\log^4 L}{L^2}\right) \ .$$

Proof. It is readily checked that

$$cum\left\{\frac{\widehat{C}_l}{C_l},\frac{\widehat{C}_l}{C_l},\frac{\widehat{C}_l}{C_l},\frac{\widehat{C}_l}{C_l}\right\} = O(l^{-3}) \ ,.$$

$$
\begin{aligned}
& cum\left\{\frac{\widehat{G}_L(\alpha_0)}{G_0}, \frac{\widehat{G}_L(\alpha_0)}{G_0}, \frac{\widehat{G}_L(\alpha_0)}{G_0}, \frac{\widehat{G}_L(\alpha_0)}{G_0}\right\} \\
= & \frac{1}{L^8}\sum_l (2l+1)^4 cum\left\{\frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}\right\} = O(L^{-6}) \ .
\end{aligned}
$$

The proof can be divided into 5 cases:

1.

$$
\begin{aligned}
& \frac{1}{L^4} cum\left\{\sum_{l_1} A_{l_1}, \sum_{l_2} A_{l_2}, \sum_{l_3} A_{l_3}, \sum_{l_4} A_{l_4}\right\} \\
= & \frac{1}{L^4}\sum_l (2l+1)^4 \log^4 l cum\left\{\frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}\right\} \\
= & O(\frac{1}{L^4}\sum_l (2l+1)^2 \log^4 l) = O(\frac{\log^4 L}{L^2}) \ ;
\end{aligned}
$$

2.

$$
\begin{aligned}
& \frac{1}{L^4} cum\left\{\sum_{l_1} B_{l_1}, \sum_{l_2} B_{l_2}, \sum_{l_3} B_{l_3}, \sum_{l_4} B_{l_4}\right\} \\
= & \frac{1}{L^4}\left\{\sum_l (2l+1)\log l\right\}^4 cum\left\{\frac{\widehat{G}_L(\alpha_0)}{G_0}, \frac{\widehat{G}_L(\alpha_0)}{G_0}, \frac{\widehat{G}_L(\alpha_0)}{G_0}, \frac{\widehat{G}_L(\alpha_0)}{G_0}\right\} \\
= & \frac{1}{L^4}\left\{\sum_l (2l+1)\log l\right\}^4 \frac{1}{L^6} = O(\frac{\log^4 L}{L^2}) \ ;
\end{aligned}
$$

3.

$$
\begin{aligned}
& \frac{1}{L^4} cum\left\{\sum_{l_1} A_{l_1}, \sum_{l_2} B_{l_2}, \sum_{l_3} B_{l_3}, \sum_{l_4} B_{l_4}\right\} \\
= & \frac{1}{L^4}\left\{\sum_{l_1} (2l_1+1)\log l_1\right\}^3 \sum_{l_2}(2l_2+1)\log l_2 cum\left\{\frac{\widehat{C}_{l_2}}{C_{l_2}}, \frac{\widehat{G}_L(\alpha_0)}{G_0}, \frac{\widehat{G}_L(\alpha_0)}{G_0}, \frac{\widehat{G}_L(\alpha_0)}{G_0}\right\} \\
= & \frac{1}{L^{10}}\left\{\sum_{l_1} (2l_1+1)\log l_1\right\}^3 \sum_{l_2}(2l_2+1)\log l_2 \\
& \times cum\left\{\frac{\widehat{C}_{l_2}}{C_{l_2}}, \sum_{l_3}(2l_3+1)\frac{\widehat{C}_{l_3}}{C_{l_3}}, \sum_{l_3}(2l_4+1)\frac{\widehat{C}_{l_4}}{C_{l_4}}, \sum_{l_5}(2l_5+1)\frac{\widehat{C}_{l_5}}{C_{l_5}}\right\} \\
= & \frac{\log^3 L}{L^4}\sum_l (2l+1)^4 \log l cum\left\{\frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}\right\}
\end{aligned}
$$

$$= O(\frac{\log^3 L}{L^4}\sum_l (2l+1)\log l) = O(\frac{\log^4 L}{L^2}) \ ;$$

4.

$$\frac{1}{L^4} cum \left\{ \sum_{l_1} A_{l_1}, \sum_{l_2} A_{l_2}, \sum_{l_3} B_{l_3}, \sum_{l_4} B_{l_4} \right\}$$

$$= \frac{1}{L^4}\sum_l (2l+1)^2 \log^2 l cum \left\{ \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \sum_{l_3}(2l_3+1)\log l_3 \frac{\widehat{G}_L(\alpha_0)}{G_0}, \sum_{l_3}(2l_4+1)\log l_4 \frac{\widehat{G}_L(\alpha_0)}{G_0} \right\}$$

$$= \frac{1}{L^8}\left\{ \sum_l (2l+1)\log l \right\}^2 \sum_l (2l+1)^2 \log^2 l cum \left\{ \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \sum_{l_3}(2l_3+1)\frac{\widehat{C}_{l_3}}{C_{l_3}}, \sum_{l_4}(2l_4+1)\frac{\widehat{C}_{l_4}}{C_{l_4}} \right\}$$

$$= \frac{1}{L^8}\left\{ \sum_l (2l+1)\log l \right\}^2 \sum_l (2l+1)^4 \log^2 l cum \left\{ \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l} \right\}$$

$$= \frac{K}{L^8}\left\{ \sum_l (2l+1)\log l \right\}^2 \sum_l (2l+1)\log^2 l = O(\frac{\log^4 L}{L^2}) \ ;$$

5.

$$\frac{1}{L^4} cum \left\{ \sum_{l_1} A_{l_1}, \sum_{l_2} A_{l_2}, \sum_{l_3} A_{l_3}, \sum_{l_4} B_{l_4} \right\}$$

$$= \frac{1}{L^4}\sum_l (2l+1)^3 \log^3 l cum \left\{ \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \sum_{l_1}(2l_1+1)\log l_1 \frac{\widehat{G}_L(\alpha_0)}{G_0} \right\}$$

$$= \frac{1}{L^6}\left\{ \sum_{l_1} (2l_1+1)\log l_1 \right\} \sum_l (2l+1)^3 \log^3 l cum \left\{ \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \sum_{l_2}(2l_2+1)\frac{\widehat{C}_{l_2}}{C_{l_2}} \right\}$$

$$= \frac{1}{L^6}\left\{ \sum_{l_1} (2l_1+1)\log l_1 \right\} \sum_l (2l+1)^4 \log^3 l cum \left\{ \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l}, \frac{\widehat{C}_l}{C_l} \right\}$$

$$= \frac{1}{L^6}\left\{ \sum_{l_1} (2l_1+1)\log l_1 \right\} \sum_l (2l+1)\log^3 l = O(\frac{\log^4 L}{L^2}) \ .$$

■

References

[1] **Anderes, E., (2010)** On the Consistent Separation of Scale and Variance for Gaussian Random Fields, *Annals of Statistics,* 38, no. 2, 870–893

[2] **Baldi, P. and Marinucci, D. (2007)** Some Characterizations of the Spherical Harmonics Coefficients for Isotropic Random Fields, *Statistics and Probability Letters*, 77, 490–496

[3] **Baldi, P., Kerkyacharian, G., Marinucci, D. and Picard, D. (2009)** Asymptotics for Spherical Needlets, *Annals of Statistics,* Vol. 37, No. 3, 1150-1171

[4] **Baldi, P., Kerkyacharian, G., Marinucci, D. and Picard, D. (2009)** Subsampling Needlet Coefficients on the Sphere, *Bernoulli*, Vol. 15, 438-463, arXiv: 0706.4169

[5] **Bingham, N.H., Goldie, C.M., Teugels, J.L. (1987)** *Regular variation*, Encyclopedia of Mathematics and its Applications, 27. Cambridge University Press, Cambridge

[6] **Brillinger, D.R. (1975)** Statistical inference for stationary point processes, in *Stochastic processes and related topics,* Proc. Summer Res. Inst. Statist. Inference for Stochastic Processes, Indiana Univ., Bloomington, Indiana..

[7] **Cabella, P. and Marinucci, D. (2009)** Statistical Challenges in the Analysis of Cosmic Microwave Background Radiation, *Annals of Applied Statistics,* Vol.2, pp.61-95

[8] **Dodelson, S. (2003)** *Modern Cosmology*, Academic Press

[9] **Durastanti, C., Geller, D., Marinucci, D., (2011),** Adaptive Nonparametric Regression of Spin Fiber Bundles on the Sphere, *Journal of Multivariate Analysis*, Vol. 104, Issue 1, pp. 16-38

[10] **Durastanti, C. (2011)** *Semiparametric and Nonparametric Estimation on the Sphere by Needlet Methods*, PhD thesis.

[11] **Faÿ, G., F. Guilloux, M. Betoule, J.-F. Cardoso, J. Delabrouille, M. Le Jeune (2008),** CMB Power Spectrum Estimation Using Wavelets, *Physical Review D,* D78:083013

[12] **Geller, D., Lan, X. and Marinucci, D. (2009)** Spin Needlets Spectral Estimation, *Electronic Journal of Statistics*, Vol. 3, 1497-1530

[13] **Hamann, J. and Wong, Y. Y. Y. (2008)** The Effects of Cosmic Microwave Background (CMB) Temperature Uncertainties on Cosmological Parameter Estimation, *Journal of Cosmology and Astroparticle Physics,* Issue 03, 025

[14] **Ivanov, A.V. and Leonenko, N. (1989)** Statistical Analysis of Random Fields, Mathematics and its Applications (Soviet Series), 28. Kluwer Academic Publishers, Dordrecht

[15] **Kerkyacharian, G., Pham Ngoc, T.M., Picard, D. (2011),** Localized Spherical Deconvolution, *Annals of Statistics,* Vol. 39, no. 2, 1042–1068

[16] **Kim, P.T., Koo, J.-Y. (2002)** Optimal Spherical Deconvolution, *Journal of Multivariate Analysis*, 80, 21-42

[17] **Kim, P.T., Koo, J.-Y., Luo, Z.-M. (2009)** Weyl Eigenvalue Asymptotics and Sharp Adaptation on Vector Bundles, *Journal of Multivariate Analysis*, 100, 1962-1978

[18] **Koo, J.-Y. and Kim, P.T. (2008)** Sharp Adaptation for Spherical Inverse Problems with Applications to Medical Imaging, *Journal of Multivariate Analysis*, 99, 165-190

[19] **Lan, X. and Marinucci, D. (2008)** The Needlets Bispectrum, *Electronic Journal of Statistics*, Vol. 2, pp.332-367

[20] **Larson, D. et al. (2011)** Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Power Spectra and WMAP-Derived Parameters, *Astrophysical Journal Supplement Series,* 192:16.

[21] **Leonenko, N. (1999)** *Limit Theorems for Random Fields with Singular Spectrum,* Mathematics and its Applications, 465. Kluwer Academic Publishers, Dordrecht

[22] **Leonenko, N. and Sakhno L. (2009)** On Spectral Representations of Tensor Random Fields on the Sphere, *Stochastic Analysis and its Applications,* in press, arXiv:0912.3389

[23] **Loh, W.-L. (2005)** Fixed-Domain Asymptotics for a Subclass of Matérn-type Gaussian Random Fields, *Annals of Statistics*, Vol. 33, No. 5, 2344-2394

[24] **Malyarenko, A. (2009)**, Invariant Random Fields in Vector Bundles and Applications to Cosmology, arXiv: 0907.4620

[25] **Marinucci, D. and Peccati, G. (2010),** Ergodicity and Gaussianity for Spherical Random Fields. *Journal of Mathematical Physics* 51, 043301

[26] **Marinucci, D. and Peccati, G.(2011)** *Random Fields on the Sphere. Representation, Limit Theorem and Cosmological Applications*, Cambridge Univ. Pr.

[27] **Marinucci, D., Pietrobon, D., Balbi, A., Baldi, P., Cabella, P., Kerkyacharian, G., Natoli, P. Picard, D., Vittorio, N., (2008)** Spherical Needlets for CMB Data Analysis, *Monthly Notices of the Royal Astronomical Society*, Volume 383, Issue 2, pp. 539-545

[28] **Mhaskar, H.N., Narcowich, F.J., Petrushev, P. and Ward, J.D. (2000),** Spherical Marcinkiewicz-Zygmund Inequalities and Positive Quadrature, *Math. Comp. 70 (2001),* no. 235, 1113-1130

[29] **Narcowich, F.J., Petrushev, P. and Ward, J.D. (2006a)** Localized Tight Frames on Spheres, *SIAM Journal of Mathematical Analysis* Vol. 38, pp. 574–594

[30] **Newey, W.K and McFadden, D. (1986)** Large Sample Estimation and Hypothesis Testing, Ch. 36, *Handbook of Econometrics*, North-Holland.

[31] **Nourdin, I. and Peccati, G. (2009)** Stein's Method on Wiener Chaos, *Probability Theory and Related Fields,* 145, no. 1-2, 75–118.

[32] **Pietrobon, D., Balbi, A., Marinucci, D. (2006)** Integrated Sachs-Wolfe Effect from the Cross Correlation of WMAP3 Year and the NRAO VLA Sky Survey Data: New Results and Constraints on Dark Energy, *Physical Review D,* id. D:74, 043524

[33] **Pietrobon, D., Amblard, A., Balbi, A., Cabella, P., Cooray, A., Marinucci, D. (2008)** Needlet Detection of Features in WMAP CMB Sky and the Impact on Anisotropies and Hemispherical Asymmetries, *Physical Review D*, D78 103504

[34] **Robinson, P.M. (1995),** Gaussian Semiparametric Estimation for Long Range Dependence, *The Annals of Statistics,* 22, 1630-1661.

[35] **Stein, E.M. and Weiss, G. (1971)** *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton University Press

[36] **Wang, D. and Loh, W.-L. (2011)** , On Fixed-Domain Asymptotics and Covariance Taperong in Gaussian Random Field Models, *Electronic Journal of Statistics*, Vol. 5, 238-269